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Influence of the regularity of the test functions for weak convergence in numerical discretization of SPDEs[☆]

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ABSTRACT

This article investigates the role of the regularity of the test function when considering the weak error for standard spatial and temporal discretizations of SPDEs of the form $dX(t) = AX(t)dt + dW(t)$, driven by space–time white noise. In previous results, test functions are assumed (at least) of class $\mathcal{C}^2$ with bounded derivatives, and the weak order is twice the strong order.

We prove that to quantify the speed of convergence, it is crucial to control some derivatives of the test functions, even if the noise is non-degenerate. First, the supremum of the weak error over all bounded continuous functions, which are bounded by 1, does not converge to 0 as the discretization parameter vanishes. Second, when considering bounded Lipschitz test functions, the weak order of convergence is divided by 2, *i.e.* it is not better than the strong order.

This is in contrast with the finite dimensional case, where the Euler–Maruyama discretization of elliptic SDEs $dY(t) = f(Y(t))dt + dB_t$ has weak order of convergence 1 even for bounded continuous functions.

1. Introduction

The numerical analysis of Stochastic Partial Differential Equations (SPDEs) has received a lot of attention in the last two decades, see for instance the recent monographs [19,24,26]. Many temporal

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and spatial discretization schemes have been studied in the literature: Euler schemes, exponential Euler schemes, and spectral Galerkin methods, Finite Element methods.

In this article, we consider linear, parabolic, equations, with additive, space–time white noise, of the type

$$\begin{cases} dX = \partial_{\xi\xi} X dt + dW, \ t > 0, \ \xi \in (0,1), \\ X(0,t) = X(1,t) = 0, \\ X(\xi, 0) = x(\xi), \end{cases}$$

on the interval $(0, 1)$, with homogeneous Dirichlet boundary conditions. More precisely, we consider Hilbert-space valued stochastic processes, which are solutions in $H = L^2(0, 1)$ of

$$dX(t) = AX(t)dt + dW(t), \ X(0) = x, \tag{1}$$

in the framework of [12], see Eq. (6) and Section 2 for precise assumptions. The noise is given by a cylindrical Wiener process, which is a mathematical model for Gaussian space–time white noise.

The extension of the results to the case of semilinear equations

$$dX(t) = AX(t)dt + F(X(t))dt + dW(t),$$

with sufficiently regular nonlinear operator F, is straightforward and is thus not considered with details in this article.

We are interested in weak convergence rates for numerical approximations of $X(T)$, for arbitrary time $T \in (0, \infty)$. Recall that this notion corresponds to studying the weak error

$$\mathbb{E}[\phi(X(T))] - \mathbb{E}[\phi(X_h(T))], \tag{2}$$

where $X_h(T)$ is the numerical approximation of $X(T)$, obtained by temporal and/or spatial discretization of the equation (with discretization parameter $h \to 0$), and $\phi : H \to \mathbb{R}$ is a bounded continuous function. Test functions with polynomial growth may also be considered. Recall also that strong convergence refers to the analysis of the strong error

$$\mathbb{E}|X(T) - X_h(T)|.$$

These notions have been extensively studied in the case of Stochastic Differential Equations (SDEs) of the type

$$dY_t = f(Y_t)dt + \sigma(Y_t)dB_t, \ Y_0 = y \in \mathbb{R}^d, \tag{3}$$

with smooth coefficients f and σ, and a d-dimensional Brownian Motion B, see for instance the classic monographs [21,27].

Strong convergence for discretizations of the SPDEs (1), also with multiplicative noise perturbation, has been studied, for instance, in [13,17,19,28,31,34] (the list is not exhaustive). Results concerning weak convergence rates have essentially been obtained in the last decade, using different approaches. In the case of the SPDE (1), which is linear with additive noise, see [15,16,22,23]. For semilinear equations, see [2,6,14,30,32,33], for an approach related to the Kolmogorov equation. See [11,18,20], where a mild Itô formula is used. Finally, for semilinear equations with additive noise, see [1] and [7] for different approaches. Deriving weak convergence rates is fundamental in infinite dimension, see for instance [25]. Moreover, it is the appropriate notion for the approximation of invariant distribution (in the asymptotic regime $T \to \infty$), see [5,8,9]. The extension of the results of this article in this long time regime is straightforward.

The results in the references mentioned above can be roughly summarized as follows: if the strong error converges with order r, then the weak error converges with order $2r$, for functions ϕ which are sufficiently smooth, *i.e.* of class $\mathcal{C}^p$, bounded and with bounded derivatives of order $1, \ldots, p$, with $p \geq 2$ (p depends on the model, for instance whether noise is additive or multiplicative):

$$\mathbb{E}|X(T) - X_h(T)| \leq C(T)h^r \ , \quad \left|\mathbb{E}[\phi(X(T))] - \mathbb{E}[\phi(X_h(T))]\right| \leq C(T)\|\phi\|_p h^{2r}, \tag{4}$$

where $\|\phi\|_p = \sup_{x\in H}|\phi(x)| + \sum_{j=1}^{p} \sup_{x\in H}|D^j\phi(x)|$. For spectral Galerkin discretization of the SPDE (1), in dimension N, with $h = \frac{1}{N}$, one may choose $r \in [0, \frac{1}{2})$. For linear implicit Euler discretization of (1), with time step size $h = \Delta t$, one may choose $r \in [0, \frac{1}{4})$.

In the SDE case, consider the Euler–Maruyama discretization of (3) (see (16)), with time step size h. Under an appropriate hypoellipticity assumption (which is satisfied in the additive non-degenerate noise case $\sigma(x) = \mathrm{Id}$), using Malliavin calculus techniques and regularization effect in the associated Kolmogorov equation, the authors in [3,4] (see also [10]), have proved that the standard approach of [29], to prove the weak error estimate for sufficiently regular functions,

$$|\mathbb{E}[\phi(Y(T))] - \mathbb{E}[\phi(Y_h(T))]| \leq C(T)\|\phi\|_p h$$

with $p \geq 2$, can be extended with $\|\phi\|_0$ instead of $\|\phi\|_p$ on the right-hand side. In other words, weak convergence is also of order 1 when considering bounded continuous (or even only measurable) test functions.

In the SPDE case, the situation is quite different, when the spatial discretization is performed by the spectral Galerkin method, and the temporal discretization is performed by the linear implicit Euler scheme. Let us formulate precisely what is the problem treated in this article. In the literature, it is standard to study the weak error (2) for a single, given, test function ϕ. However, weak error estimates of the type (4) are more relevant, since they reveal the interplay between the regularity properties of the test functions ϕ, and the weak order of convergence.

The objective of this article is to provide answers to the following question.

Question 1. *What are the values of $r \geq 0$, such that, for any $T \in (0, \infty)$, there exists $C_r(T) \in (0, \infty)$, such that, for all (sufficiently small) $h > 0$, one has*

$$\Big|\mathbb{E}[\phi(X(T))] - \mathbb{E}[\phi(X_h(T))]\Big| \leq C_r(T)\|\phi\|_p h^r,$$

for the following two classes of test functions $\phi : H \to \mathbb{R}$:

(i) bounded continuous functions?
(ii) bounded and Lipschitz continuous functions?

Let us insist that in Question 1 (and more generally in this article), the weak order of convergence r is assumed not to depend on ϕ, except on its regularity properties, *i.e.* on the integer p.

Note that only bounded test functions, with bounded derivatives, are considered. This is not restrictive: taking into account polynomial growth may be possible and would not change the answers provided to Question 1, and to Question 2. In addition, note that for any given bounded and continuous test function $\phi : H \to \mathbb{R}$, one has

$$\mathbb{E}[\phi(X_h(T))] \underset{h\to 0}{\to} \mathbb{E}[\phi(X(T))],$$

thanks to strong convergence.

More precisely, the following reformulation of Question 1 is considered in this article. Indeed, it is natural to consider the supremum of the error over test functions ϕ of a given regularity p, with a condition $\|\phi\|_p \leq 1$ (by linearity, this condition is not restrictive).

Question 2. *What is the order of convergence to 0, when $h \to 0$, of*

$$\sup_{\phi\in\Phi} \Big|\mathbb{E}[\phi(X(T))] - \mathbb{E}[\phi(X_h(T))]\Big|, \tag{5}$$

where

(i) $\Phi = \{\phi \in \mathcal{C}^0(H, \mathbb{R});\ \|\phi\|_0 \leq 1\}$?
(ii) $\Phi = \{\phi \in \mathcal{C}^0(H, \mathbb{R});\ \|\phi\|_0 + \mathrm{Lip}(\phi) \leq 1\}$?

Above, the Lipschitz semi-norm is defined as $\mathrm{Lip}(\phi) = \sup_{x_1\neq x_2} \frac{|\phi(x_2)-\phi(x_1)|}{|x_2-x_1|}$.

Questions 1 and 2 are equivalent, since the orders of convergence in Question 1 are required not to depend on ϕ, but only on its regularity properties.

On the one hand, considering the supremum in (5) is convenient for the proofs below: instead of exhibiting a single test function which would correspond to the worst case scenario for the weak order of convergence in the given class of test function, the proof will use well-chosen parametrized families, and appropriate limiting arguments to which make the relevant norms blow up.

On the other hand, the quantity (5) can be interpreted in terms of the Radon distance and of the Wasserstein 1 distance between probability distributions, *i.e.* the laws of $X(T)$ and $X_h(T)$. This was not the initial motivation of this work, and this point of view will not be discussed with details.

Note that the noise in the SPDE (1) is non-degenerate (the covariance operator is the identity). With intuition built upon the SDE case, it might be expected that the order of convergence in Question 2 does not depend on p, and is equal to the same order as when $p = 2$, *i.e.* $r \in [0, 1)$ if $h = \frac{1}{N}$ (spectral Galerkin discretization in dimension N) or $r \in [0, \frac{1}{2})$ if $h = \Delta t$ (linear implicit Euler scheme, with time step size Δt).

However, maybe surprisingly, that intuition leads to uncorrect results. The main contribution of this article is to provide the following answer to Question 2. First, in the case (*i*), then (5) does not converge to 0 when $h \to 0$. Second, in the case (*ii*), then the order of convergence for (5) is equal to the associated strong order, *i.e.* one needs $r \in [0, \frac{1}{2})$ if $h = \frac{1}{N}$ (spectral Galerkin discretization in dimension N) or $r \in [0, \frac{1}{4})$ if $h = \Delta t$ (linear implicit Euler scheme, with time step size Δt). Precise statements, Theorems 1 and 2, are provided in Section 3.

Those answers to Question 2 lead to the following conclusion: in general, the regularity of the test functions, and the control of derivatives, is essential to quantify the speed of convergence of the weak error (2) for numerical discretizations of SPDEs (1).

The proofs below use well-chosen families of functions which have low significance for concrete numerical approximation. It may be possible to define smaller families of non-regular test functions, for which uniform convergence of the numerical schemes holds true with better rates of convergence. This is expected to be obtained by the generalization of the finite dimensional approach of [3,4]: regularization effect in the Kolmogorov equation and Malliavin calculus techniques. The identification of the appropriate setting is left for future works.

Why the regularity of the test functions matters for SPDEs may be explained by the properties of the solutions of associated Kolmogorov equations. Indeed, as emphasized in [2,6,14], Sobolev-type regularity properties for the spatial derivatives of the solution of this infinite dimensional PDE are required to treat the most irregular terms in the error expansion. Similar arguments appear in [11,15] and related articles. The regularity estimates have singularities at the initial time, even when the test function (seen as the initial condition of the Kolmogorov equation) is regular.

For SDEs, the Kolmogorov equation preserves regularity of the initial condition. Singularities only appear when a regularization effect is needed, in an hypoelliptic setting.

For SPDEs, exhibiting a rate of convergence in the error analysis is only possible when using some spatial regularity property, as mentioned above. The better the spatial regularity, the greater the order of convergence, but the stronger the singularity — with the constraint of remaining integrable. This approach yields the optimal order of convergence for regular test functions. Weakening the regularity condition on the test functions then introduces even stronger singularities, and less spatial regularity may be used: in turn the order of convergence decreases. The optimality of these heuristic arguments is validated by Theorems 1 and 2.

The article is organized as follows. Assumptions on the model and numerical discretization schemes are introduced in Section 2. Section 2.3 describes important spatial regularity properties, which are very different for the discretized versions, compared with the exact solution. Our main results, Theorem 1 (bounded continuous functions) and Theorem 2 are stated in Section 3. Detailed proofs are provided in Section 4.

2. Setting

2.1. Model and assumptions

The model in this article is given by a Stochastic Partial Differential Equation (SPDE),

$$dX(t) = AX(t)dt + dW(t), \quad X(0) = 0, \tag{6}$$

i.e. by Eq. (3), with the initial condition set equal to 0. Extending the results of this article to arbitrary initial conditions is straightforward.

2.1.1. Linear operator A

Denote by $\langle\cdot,\cdot\rangle$, resp. $|\cdot|$, the inner product, resp. the norm, in the separable Hilbert space $H = L^2(0,1)$.

The operator A in the SPDE is the Laplace operator with homogeneous Dirichlet boundary conditions, and thus it satisfies the following conditions.

Assumption 1. The mapping A is an unbounded, self-adjoint, linear operator on H.

Define, for all $n \in \mathbb{N} = \{1, \ldots\}$,

$$\lambda_n = \pi^2 n^2 \ , \quad e_n = \sqrt{2}\sin\big(n\pi\,\cdot\big).$$

Then the operator A and its domain $D(A)$ are given by

$$Ax = \sum_{n\in\mathbb{N}} -\lambda_n \langle x, e_n\rangle \ , \quad \forall\, x \in D(A) = \Big\{ x \in H \ ; \ \sum_{n\in\mathbb{N}} \lambda_n^2 \langle x, e_n\rangle^2 < \infty \Big\}.$$

Recall that $\big(e_n\big)_{n\in\mathbb{N}}$ is a complete orthonormal system of H.

Introduce the following notation.

Definition 1.

(1) The operator A generates a strongly-continuous semigroup $\big(e^{tA}\big)_{t\ge 0}$ on H, with

$$e^{tA}x = \sum_{n\in\mathbb{N}} e^{-\lambda_n t}\langle x, e_n\rangle \ , \quad \forall\, x \in H, t \ge 0.$$

(2) For all $\alpha \in [0,1]$, set

$$|x|_\alpha = \Big(\sum_{n\in\mathbb{N}} \lambda_n^{2\alpha}\langle x, e_n\rangle^2\Big)^{\frac{1}{2}} \in [0,\infty], \qquad \forall\, x \in H.$$

2.1.2. Cylindrical Wiener process

Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ denote a probability space. The expectation operator is denoted by $\mathbb{E}$.

Assumption 2. Let $\big(\beta_n\big)_{n\in\mathbb{N}}$ be a sequence of independent standard $\mathbb{R}$-valued Wiener processes. Then set, for all $t \ge 0$,

$$W(t) = \sum_{n\in\mathbb{N}} \beta_n(t) e_n. \tag{7}$$

It is a standard fact that, for all $t \ge 0$, almost surely the series in (7) does not converge in H. However, if $\Phi \in \mathcal{L}(H)$ is an Hilbert–Schmidt operator, then $\Phi W(t) = \sum_{n\in\mathbb{N}} \beta_n(t)\Phi e_n$ is a Wiener process in H, with covariance operator $\Phi\Phi^\star$.

2.1.3. Mild solution

Solutions of the SPDE (6) are interpreted in the mild sense: the unique solution, which is often called stochastic convolution, is given by

$$X(t) = \int_0^t e^{(t-s)A} dW(s) = \sum_{n\in\mathbb{N}} \Big(\int_0^t e^{-\lambda_n(t-s)} d\beta_n(s)\Big) e_n, \quad t \in (0,\infty).$$

With the notation $X_n(t) = \langle X(t), e_n\rangle = \int_0^t e^{-\lambda_n(t-s)} d\beta_n(s)$, the process $\big(X_n(t)\big)_{t\ge 0}$ is independent Ornstein–Uhlenbeck processes. Thus, $\big(X(t)\big)_{t\ge 0}$ is a centered Gaussian process with values in H. Let

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μ_t denote the law of $X(t)$, *i.e.* the centered Gaussian probability distribution on H with covariance operator $Q_t \in \mathcal{L}(H)$, given by

$$Q_t e_n = \frac{1}{2\lambda_n}\big(1 - e^{-2\lambda_n t}\big)e_n = \mathbb{E}\big[|X_n(t)|^2\big]e_n,$$

for all $n \in \mathbb{N}$ and $t \geq 0$.

2.2. Numerical schemes

Space and time discretization schemes are defined below. One may also consider full-discretization schemes obtained by combining these two procedures.

2.2.1. Space discretization: spectral Galerkin method

For every $N \in \mathbb{N}$, let $P_N \in \mathcal{L}(H)$ denote the orthogonal projection onto the finite-dimensional subspace $\mathrm{Span}\big(e_1, \ldots, e_N\big)$:

$$P_N x = \sum_{n=1}^{N} \langle x, e_n \rangle e_n, \quad \forall\, x \in H.$$

The process $X^{(N)}$ obtained by discretization in space of the SPDE (6), is solution of

$$dX^{(N)}(t) = AX^{(N)}(t)dt + P_N dW(t)\ , \quad X^{(N)}(0) = 0.$$

In fact, $X^{(N)}(t) = P_N X(t)$, for all $t \geq 0$ and $N \in \mathbb{N}$.

Let then $\mu_t^{(N)}$ denote the law of the random variable $X^{(N)}$: it is a centered Gaussian probability distribution, with covariance operator $P_N Q_t (P_N)^\star = P_N Q_t$.

2.2.2. Time discretization: linear implicit Euler scheme

Let $\Delta t > 0$ denote a time-step size, without restriction we assume $\Delta t \in (0, 1)$. The scheme is defined such that for all $k \in \mathbb{N}_0 = \{0, 1, \ldots\}$,

$$X_{k+1}^{\Delta t} = X_k^{\Delta t} + \Delta t A X_{k+1}^{\Delta t} + \Delta W_k\ , \quad X_0^{\Delta t} = 0,$$

with Wiener increments $\Delta W_k = W\big((k+1)\Delta t\big) - W\big(k\Delta t\big)$.

Rigorously,

$$X_{k+1}^{\Delta t} = S_{\Delta t} X_k + S_{\Delta t} \Delta W_k,$$

where $S_{\Delta t} = \big(I - \Delta t A\big)^{-1}$ is a linear self-adjoint Hilbert–Schmidt operator on H.

As a consequence, for every $k \in \mathbb{N}$,

$$X_k^{\Delta t} = \sum_{\ell=0}^{k-1} S_{\Delta t}^{k-\ell} \Delta W_\ell,$$

and the law $\nu_k^{\Delta t}$ of $X_k^{\Delta t}$ is a centered Gaussian probability distribution, with covariance operator

$$Q_k^{\Delta t} = \Delta t \sum_{\ell=0}^{k-1} S_{\Delta t}^{2(k-\ell)}.$$

2.3. Space regularity properties

The aim of this section is to provide some important results concerning the moments $\int_H |x|_\alpha^2 \mu(dx)$, for different values of $\alpha \in [0, 1]$. The parameter α is interpreted as indicating space regularity of the process. We emphasize on the key observation: the behaviors are different when considering, on the one hand, $\mu = \mu_t$, and, on the other hand, $\mu = \mu_t^{(N)}$ or $\mu = \nu_n^{\Delta t}$, which are obtained by the discretization schemes. We will take advantage of this property in the study of the orders of convergence for bounded continuous test functions.

First, consider the law μ_t at time t, of the solution of the SPDE (6): for any $t \in (0, \infty)$,

$$\int_H |x|_\alpha^2 \mu_t(dx) = \sum_{n\in\mathbb{N}} \frac{1}{2\lambda_n^{1-2\alpha}}\big(1 - e^{-2\lambda_n t}\big) < \infty \iff \alpha \in [0, \frac{1}{4}). \tag{8}$$

Now, consider the law $\mu_t^{(N)}$, at time t, obtained by spatial discretization: for every $t \in [0, \infty)$,

$$\begin{aligned} &\int_H |x|_\alpha^2 \mu_t^{(N)}(dx) < \infty, \quad \forall\, \alpha \in [0, 1],\ \forall\, N \in \mathbb{N},\\ &\sup_{N\in\mathbb{N}} \int_H |x|_\alpha^2 \mu_t^{(N)}(dx) < \infty \iff \alpha \in [0, \frac{1}{4}). \end{aligned} \tag{9}$$

Finally, consider the law $\nu_k^{\Delta t}$, at time k, obtained by the temporal discretization: for every $k \in \mathbb{N}$ and every $T \in (0, \infty)$,

$$\begin{aligned} &\int_H |x|_\alpha^2 \nu_k^{\Delta t}(dx) < \infty, \quad \forall\, \alpha \in [0, \frac{3}{4}),\ \forall\, \Delta t \in (0, 1),\ \forall\, k \in \mathbb{N},\\ &\sup_{\Delta t\in(0,1)} \int_H |x|_\alpha^2 \nu^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor}(dx) < \infty \iff \alpha \in [0, \frac{1}{4}). \end{aligned} \tag{10}$$

Observe that in (9) and (10), one recovers the same behavior as in (8), only when the supremum over all discretization parameters ($N \in \mathbb{N}$ and $\Delta t \in (0, 1)$) is computed. For fixed values of these parameters, some larger values of $\alpha \geq \frac{1}{4}$ are allowed.

The proofs of estimates in (8) and (9) are straightforward. For completeness, let us give a detailed proof of the estimates in (10). Similar arguments will be used again below.

To prove the first statement in (10), let $\Delta t > 0$, $k \in \mathbb{N}$, and $\alpha \in [0, 1]$, then

$$\begin{aligned} \int_H |x|_\alpha^2 \nu_k^{\Delta t}(dx) &= \sum_{n\in\mathbb{N}} \lambda_n^{2\alpha} \langle Q_k^{\Delta t} e_n, e_n\rangle = \Delta t \sum_{n\in\mathbb{N}} \lambda_n^{2\alpha} \sum_{\ell=1}^{k} \frac{1}{(1+\lambda_n \Delta t)^{2\ell}}\\ &= \sum_{n\in\mathbb{N}} \frac{\lambda_n^{2\alpha}}{\lambda_n(2+\lambda_n\Delta t)}\big(1 - \frac{1}{(1+\lambda_n\Delta t)^{2k}}\big)\\ &< \infty \iff \alpha \in [0, \frac{3}{4}). \end{aligned}$$

To prove the second statement, first assume $\alpha \in [0, \frac{1}{4})$, then $1 - 2\alpha > \frac{1}{2}$, thus for all $\Delta t \in (0, 1)$,

$$\int_H |x|_\alpha^2 \nu^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor}(dx) \leq \sum_{n\in\mathbb{N}} \frac{1}{2\lambda_n^{1-2\alpha}} < \infty.$$

Now assume that $\alpha \geq \frac{1}{4}$. By a monotonicity argument, it is sufficient to consider the case $\alpha = \frac{1}{4}$. Let $M \in \mathbb{N}$ be an auxiliary integer, and choose $\Delta t = \frac{T}{N^2}$, with $N \in \mathbb{N}$, $N \geq M$.

$$\begin{aligned} \int_H |x|_{\frac{1}{4}}^2 \nu_{N^2}^{\frac{T}{N^2}}(dx) &= \sum_{n\in\mathbb{N}} \frac{1}{\pi n(2+\pi^2 T\frac{n^2}{N^2})}\big(1 - \frac{1}{(1+\pi^2 T\frac{n^2}{N^2})^{2\lfloor TN^2\rfloor}}\big)\\ &\geq \frac{1}{N} \sum_{n\geq \frac{N}{M}} \frac{1}{\pi\frac{n}{N}(2+\pi^2 T\frac{n^2}{N^2})}\big(1 - \frac{1}{(1+\pi^2 T\frac{n^2}{N^2})^{2\lfloor TN^2\rfloor}}\big). \end{aligned}$$

Note that for $n \geq \frac{N}{M}$, one has $\frac{1}{(1+\pi^2 T\frac{n^2}{N^2})^{2\lfloor TN^2\rfloor}} \leq \frac{1}{(1+\frac{\pi^2 T}{M^2})^{2\lfloor TN^2\rfloor}} \underset{N\to\infty}{\to} 0$. Then, by a Riemann sum argument, one obtains

$$\liminf_{N\to\infty} \int_H |x|_{\frac{1}{4}}^2 \nu_{N^2}^{\frac{1}{N^2}}(dx) \geq \int_{\frac{1}{M}}^{\infty} \frac{1}{\pi z(2+\pi^2 T z^2)} dz.$$

Letting $M \to \infty$ and using $\int_0^\infty \frac{1}{\pi z(2+\pi^2 T z^2)} dz = \infty$, one obtains

$$\sup_{\Delta t \in (0,1)} \int_H |x|_{\frac{1}{4}}^2 \nu^{\Delta t}_{\lfloor \frac{T}{\Delta t} \rfloor}(dx) = \infty.$$

This concludes the proof of the equivalence statement (10).

3. Main results

Introduce the following notation:

- $\|\phi\|_0 = \sup_{x \in H} |\phi(x)|$, for $\phi \in \mathcal{C}^0(H, \mathbb{R})$, bounded and continuous functions from H to $\mathbb{R}$,
- $\|\phi\|_1 = \|\phi\|_0 + \sup_{x,y \in H, x \neq y} \frac{|\phi(y)-\phi(x)|}{|y-x|}$, for $\phi \in \mathcal{C}^{0,1}(H, \mathbb{R})$, bounded and Lipschitz continuous functions from H to $\mathbb{R}$,
- $\|\phi\|_2 = \sup_{x \in H} |\phi(x)| + \sup_{x \in H, h \in H, |h| \leq 1} |D\phi(x).h| + \sup_{x \in H, h_1, h_2 \in H, |h_1| \leq 1, |h_2| \leq 1} |D^2\phi(x).(h_1, h_2)|$, for $\phi \in \mathcal{C}^2(H, \mathbb{R})$, bounded functions from H to $\mathbb{R}$ of class $\mathcal{C}^2$, with bounded first and second order derivatives.

3.1. Statements

The main result of this article is Theorem 1, which may be interpreted as follows: there is no rate of convergence to 0, for the weak error, when considering the supremum over all bounded and continuous functions.

Theorem 1. *Let $T \in (0, \infty)$. Then*

$$\begin{aligned} &\limsup_{N \to \infty} \sup_{\phi \in \mathcal{C}^0(H,\mathbb{R}), \|\phi\|_0 \leq 1} \Big| \int \phi d\mu_T - \int \phi d\mu_T^{(N)} \Big| > 0, \\ &\limsup_{\Delta t \to 0} \sup_{\phi \in \mathcal{C}^0(H,\mathbb{R}), \|\phi\|_0 \leq 1} \Big| \int \phi d\mu_T - \int \phi d\nu^{\Delta t}_{\lfloor \frac{T}{\Delta t} \rfloor} \Big| > 0. \end{aligned} \tag{11}$$

The proof of Theorem 1 is postponed to Section 4.1.

To explain why the statement of Theorem 1 may be surprising, recall that strong convergence results, with order in $[0, \frac{1}{4})$, are available: for every $r \in [0, \frac{1}{4})$, and every $T \in (0, \infty)$,

$$\limsup_{N \to \infty} \lambda_N^{2r} \mathbb{E}\big|X(T) - X^{(N)}(T)\big|^2 < \infty \ , \quad \limsup_{\Delta t \to 0} \frac{1}{\Delta t^{2r}} \mathbb{E}\big|X(T) - X^{\Delta t}_{\lfloor \frac{T}{\Delta t} \rfloor}\big|^2 < \infty. \tag{12}$$

Thus, for any bounded and continuous function $\phi \in \mathcal{C}^0(H, \mathbb{R})$, the convergence below is valid:

$$\int \phi d\mu_T^{(N)} \underset{N \to \infty}{\to} \int \phi d\mu_T \quad , \quad \int \phi d\nu^{\Delta t}_{\lfloor \frac{T}{\Delta t} \rfloor} \underset{\Delta t \to 0}{\to} \int \phi d\mu_T.$$

However, the supremum of the error over all bounded continuous functions, bounded by 1, does not converge to 0.

As will become clear in the proof of Theorem 1, see the stronger statement (18) below, the issue is not the regularity of the functions ϕ – smooth functions are used – but the lack of control of the growth of the derivatives.

It is also worth mentioning that if one considers the set of bounded measurable test functions, instead of continuous test functions, in (11), the result is straightforward. Indeed, this corresponds to looking at the total variation distance between μ_T, and $\mu_T^{(N)}$ of $\nu^{\Delta t}_{\lfloor \frac{T}{\Delta t} \rfloor}$, and due to the results of Section 2.3, these distributions are singular.

We also prove the following statement, Theorem 2, which may be interpreted as follows: the best order of convergence, for the weak error, when considering the supremum over all bounded and Lipschitz continuous functions, is equal to the strong order of convergence.

Theorem 2. *Let $T \in (0,\infty)$. Then*

$$\begin{aligned}
&\limsup_{N\to\infty} \lambda_N^r \sup_{\phi\in\mathcal{C}^{0,1}(H,\mathbb{R}),\|\phi\|_1\le 1} \Big|\int \phi d\mu_T - \int \phi d\mu_T^{(N)}\Big| = \begin{cases} 0, & \forall\, r\in[0,\frac14) \\ \infty, & \forall\, r\in(\frac14,\frac12), \end{cases} \\
&\limsup_{\Delta t\to 0} \frac{1}{\Delta t^r} \sup_{\phi\in\mathcal{C}^{0,1}(H,\mathbb{R}),\|\phi\|_1\le 1} \Big|\int \phi d\mu_T - \int \phi d\nu^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor}\Big| = \begin{cases} 0, & \forall\, r\in[0,\frac14) \\ \infty, & \forall\, r\in(\frac14,\frac12). \end{cases}
\end{aligned} \tag{13}$$

The results in Theorem 2 in the regime $r \in [0,\frac14)$ are not new, they are straightforward applications of the strong convergence estimates in (12). The case $r \in (\frac14,\frac12)$ is treated in Section 4.2.

For comparison, we state an additional result, considering test functions of class $\mathcal{C}^2$, bounded, and with bounded first and second order derivatives.

Proposition 1. *Let $T \in (0,\infty)$.*

$$\begin{aligned}
&\limsup_{N\to\infty} \lambda_N^r \sup_{\phi\in\mathcal{C}^{2}(H,\mathbb{R}),\|\phi\|_2\le 1} \Big|\int \phi d\mu_T - \int \phi d\mu_T^{(N)}\Big| = \begin{cases} 0, & \forall\, r\in[0,\frac12) \\ \infty, & \forall\, r\in(\frac12,1), \end{cases} \\
&\limsup_{\Delta t\to 0} \frac{1}{\Delta t^r} \sup_{\phi\in\mathcal{C}^{2}(H,\mathbb{R}),\|\phi\|_2\le 1} \Big|\int \phi d\mu_T - \int \phi d\nu^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor}\Big| = \begin{cases} 0, & \forall\, r\in[0,\frac12) \\ \infty, & \forall\, r\in(\frac12,1). \end{cases}
\end{aligned} \tag{14}$$

The result of Proposition 1, in the regime $r \in [0,\frac12)$, has been proved in a more general setting, for semilinear versions of (6), see the references in the introduction. In the case of multiplicative noise, the results require that ϕ is at least of class $\mathcal{C}^3$, however the order of convergence remains equal to $\frac12$ for such test functions. The case $r \in (\frac12,1)$ is obtained using the lower bounds from [11].

Note that Theorems 1 and 2 are also valid when looking at the regime $T\to\infty$, *i.e.* at the level of the invariant distributions of the process and of its discretized versions.

Comparing Theorems 1, 2 and Proposition 1 reveals that in infinite dimension, regularity of the test functions and control of derivatives play an important role in the analysis of the numerical error in the weak sense.

3.2. Comparison with the finite dimensional situation

The situation described by Theorems 1 and 2, and Proposition 1, is specific to the infinite dimensional situation. Indeed, when considering Euler–Maruyama discretization of hypoelliptic SDEs (in finite dimension), the order of convergence (equal to 1) does not change when considering either bounded continuous functions, or bounded and Lipschitz continuous functions, or functions of class $\mathcal{C}^2$.

Indeed, consider an SDE in $\mathbb{R}^d$ (see Eq. (3), with additive non-degenerate noise),

$$dY(t) = f(Y(t))dt + dB_t,\ Y(0) = y_0, \tag{15}$$

where $(B_t)_{t\ge 0}$ is a d-dimensional standard Wiener process, and $f:\mathbb{R}^d\to\mathbb{R}^d$ is a smooth bounded function, with bounded derivatives.

Consider its Euler–Maruyama discretization, with time step size $\Delta t > 0$: for $k\in\mathbb{N}_0$,

$$Y_{k+1}^{\Delta t} = Y_k^{\Delta t} + \Delta t f(Y_k^{\Delta t}) + B((k+1)\Delta t) - B(k\Delta t)\ , \quad Y_0^{\Delta t} = y_0. \tag{16}$$

The strong order of convergence in this case is equal to 1 (this is due to the fact that the noise is additive, it would be equal to $\frac12$ in general):

$$\limsup_{\Delta t\to 0} \frac{1}{\Delta t^2}\mathbb{E}|Y(T) - Y^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor}|^2 \in (0,\infty).$$

Then, it is a remarkable fact that when considering bounded continuous test functions, one still obtains an error which is of order 1, see [3,4],

$$\limsup_{\Delta t\to 0}\frac{1}{\Delta t}\sup_{\phi\in\mathcal{C}^0(H,\mathbb{R}),\|\phi\|_0\leq 1}|\mathbb{E}\phi(Y(T))-\mathbb{E}\phi(Y^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor})|\in(0,\infty),$$

for every $T\in(0,\infty)$. Equivalently, there exists $C(T)\in(0,\infty)$, such that for every bounded continuous function ϕ,

$$|\mathbb{E}\phi(Y(T))-\mathbb{E}\phi(Y^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor})|\leq C(T)\|\phi\|_0\Delta t. \tag{17}$$

Theorem 1 indicates that in infinite dimension, the generalization of (17) is not valid, both for the standard and widely used time and space discretization schemes we have considered.

4. Proofs

4.1. Bounded continuous test functions: proof of Theorem 1

In fact, a slightly stronger result than Theorem 1 is proved below:

$$\limsup_{N\to\infty}\sup_{\phi\in\Phi}|\int\phi d\mu_T-\int\phi d\mu_T^{(N)}|>0\ ,\quad \limsup_{\Delta t\to 0}\sup_{\phi\in\Phi}|\int\phi d\mu_T-\int\phi d\nu^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor}|>0, \tag{18}$$

where $\Phi\subset\mathcal{C}^\infty(H,\mathbb{R})$ is such that $\|\phi\|_0\leq 1$ for all $\phi\in\Phi$. In the examples given below, the functions ϕ are smooth and have bounded derivatives of any order, however only $\|\phi\|_0$ is uniformly bounded over $\Phi-$ precisely, $\sup\{\|\phi\|_1,\ \phi\in\Phi\}=\infty$.

We provide two different examples of sets Φ. The first family is constructed using the results of Section 2.3, concerning regularity properties of the discretized versions of the SPDE. The second family contains functions with arbitrarily fast oscillations, and is treated using some Riemann sums arguments. This proof is instructive, similar arguments appear for proving Theorem 2.

4.1.1. First proof

Define $\Phi^1=\{\phi^1_{\epsilon,M},\ \epsilon\in(0,1),M\in\mathbb{N}\}$, where

$$\phi^1_{\epsilon,M}(x)=\exp\big(-\epsilon|P_Mx|^2_{\frac14}\big),\quad \forall\, x\in H. \tag{19}$$

Then $\phi^1_{\epsilon,M}\in\mathcal{C}^\infty(H,\mathbb{R})$, and $\|\phi^1_{\epsilon,M}\|_0=1$. However, $\sup\big\{\|\phi\|_1,\ \phi\in\Phi^1\big\}=\infty$.

Remark 1. Observe that for all $x\in H$,

$$\lim_{\epsilon\to 0}\lim_{M\to\infty}\phi^1_{\epsilon,M}(x)=\overline{\phi^1}(x)=\mathbb{1}_{|x|_{\frac14}<\infty},$$

where $\overline{\phi^1}$ is only bounded and measurable. Then, thanks to the regularity results (8), (9) and (10) for all $N\in\mathbb{N}$ and $\Delta t\in(0,1)$,

$$\int\overline{\phi^1}d\mu_T=0\ ,\quad\int\overline{\phi^1}d\mu_T^{(N)}=\int\overline{\phi^1}d\nu^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor}=1.$$

This means that in total variance distance, the convergence of $\mu_T^{(N)}$ and of $\nu^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor}$ to μ_T does not hold true. The current proof consists in adapting this statement by an approximating argument.

Let

$$\delta^1_1(N)=\sup_{\phi\in\Phi^1}|\int\phi d\mu_T-\int\phi d\mu_T^{(N)}|,$$

$$\delta^1_2(\Delta t)=\sup_{\phi\in\Phi^1}|\int\phi d\mu_T-\int\phi d\nu^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor}|.$$

For every $N \in \mathbb{N}$, $\epsilon \in (0,1)$, letting $M \to \infty$ gives

$$\delta_1^1(N) \geq \big|\mathbb{E}[e^{-\epsilon|P_M X(T)|_{\frac{1}{4}}^2}] - \mathbb{E}[e^{-\epsilon|P_M X^{(N)}(T)|_{\frac{1}{4}}^2}]\big| \geq \big|0 - \mathbb{E}[e^{-\epsilon|X^{(N)}(T)|_{\frac{1}{4}}^2}]|,$$

where almost surely $|X^{(N)}(T)|_{\frac{1}{4}}^2 < \infty$, thanks to (9). On the contrary, using (8), $\mathbb{E}|X(T)|_{\frac{1}{4}}^2 = \infty$, and in fact almost surely $|X(T)|_{\frac{1}{4}}^2 = \infty$. More precisely,

$$\begin{aligned}
\mathbb{E}[e^{-\epsilon|P_M X(T)|_{\frac{1}{4}}^2}] &= \prod_{m=1}^{M} \mathbb{E}[e^{-\epsilon\lambda_m^{\frac{1}{2}}|\langle X(T),e_m\rangle|^2}] \\
&= \prod_{m=1}^{M}\big(1+\frac{\epsilon\sqrt{\lambda_m}}{\lambda_m}(1-e^{-2\lambda_m T})\big)^{-\frac{1}{2}} \\
&= \exp\big(-\frac{1}{2}\sum_{m=1}^{M}\log\big(1+\frac{\epsilon\sqrt{\lambda_m}}{\lambda_m}(1-e^{-2\lambda_m T})\big)\big) \\
&\underset{M\to\infty}{\to} 0.
\end{aligned}$$

Similarly,

$$\delta_2^1(\Delta t) \geq \mathbb{E}[e^{-\epsilon|X_{\lfloor\frac{T}{\Delta t}\rfloor}^{\Delta t}|_{\frac{1}{4}}^2}],$$

with $|X_{\lfloor\frac{T}{\Delta t}\rfloor}^{\Delta t}|_{\frac{1}{4}}^2 < \infty$ almost surely, thanks to (10).

Finally, letting $\epsilon \to 0$, for all $N \in \mathbb{N}$ and $\Delta t \in (0,1)$

$$\delta_1^1(N) \geq 1\,, \quad \delta_2^1(\Delta t) \geq 1.$$

Thus

$$\limsup_{N\to\infty}\sup_{\phi\in\Phi^1}|\int\phi d\mu_T - \int\phi d\mu_T^{(N)}| = \limsup_{N\to\infty}\delta_1^1(N) \geq 1,$$

$$\limsup_{\Delta t\to 0}\sup_{\phi\in\Phi^1}|\int\phi d\mu_T - \int\phi d\nu_{\lfloor\frac{T}{\Delta t}\rfloor}^{\Delta t}| = \limsup_{\Delta t\to 0}\delta_2^1(\Delta t) \geq 1,$$

hence (11) is validated. This concludes the first proof of Theorem 1.

4.1.2. Second proof

Define $\Phi^2 = \{\phi_M^2,\ M \in \mathbb{N}\}$, where

$$\phi_M^2(x) = \exp\big(i\sqrt{M}\langle\theta_M, x\rangle\big),\ \theta_M = \sum_{m=\frac{M}{2}}^{M} e_m, \quad \forall\, x \in H.$$

In this example, it is convenient to consider complex-valued functions, however it is straightforward to get rid of this issue.

Like above, $\Phi^2 \subset \mathcal{C}^\infty(H,\mathbb{C})$, $\|\phi\|_0 = 1$ for all $\phi \in \Phi^2$, and $\sup\{\|\phi\|_1,\ \phi\in\Phi^2\} = \infty$. Let

$$\delta_1^2(N) = \sup_{\phi\in\Phi^2}|\int\phi d\mu_T - \int\phi d\mu_T^{(N)}|\,, \quad \delta_2^2(\Delta t) = \sup_{\phi\in\Phi^2}|\int\phi d\mu_T - \int\phi d\nu_{\lfloor\frac{T}{\Delta t}\rfloor}^{\Delta t}|.$$

First, focus on $\delta_1^2(N)$. Observe that for $M \geq 2N+1$, $\langle\theta_M, X^{(N)}(T)\rangle = 0$ almost surely, hence $\int\phi_M^2 d\mu_T^{(N)} = 1$. On the contrary,

$$\int\phi_M^2 d\mu_T = \mathbb{E}[e^{i\sqrt{M}\langle\theta_M,X(T)\rangle}] = \exp\big(-M\sum_{m=\frac{M}{2}}^{M}\frac{1}{2\lambda_m}(1-e^{-2\lambda_m T})\big)$$

$$
\begin{aligned}
&= \exp\Big(-\frac{1}{M}\sum_{m=\frac{M}{2}}^{M}\frac{1}{2\pi^2\big(\frac{m}{M}\big)^2}+\mathrm{o}(1)\Big)\\
&\underset{M\to\infty}{\to} \exp\Big(-\int_{\frac12}^{1}\frac{1}{2\pi^2 z^2}dz\Big),
\end{aligned}
$$

using a Riemann sum argument, and $M\sum_{m=\frac{M}{2}}^{M}\frac{1}{2\lambda_m}e^{-2\lambda_m T}=\mathrm{O}\big(e^{-2\lambda_{\frac{M}{2}}T}\big)\underset{M\to\infty}{\to}0$.

As a consequence, for all $N\in\mathbb{N}$, and letting $M\to\infty$ (with $M\geq 2N+1$), one obtains

$$
\delta_1^2(N)\geq 1-\exp\Big(-\int_{\frac12}^{1}\frac{1}{2\pi^2 z^2}dz\Big)>0.
$$

Second, focus on $\delta_2^2(\Delta t)$. In order to use a Riemann sum argument, it is convenient to choose $\Delta t=\frac{T}{M^2}$, and to write

$$
\delta_2^2\Big(\frac{T}{M^2}\Big)\geq\Big|\int\phi_M^2 d\mu_T-\int\phi_M^2 d\nu_{M^2}^{\Delta t}\Big|,
$$

with $\int\phi_M^2 d\mu_T\underset{M\to\infty}{\to}\exp\big(-\int_{\frac12}^{1}\frac{1}{2\pi^2z^2}dz\big)$, as above, and

$$
\begin{aligned}
\int\phi_M^2 d\nu_{M^2}^{\Delta t}&=\mathbb{E}\big[\exp\big(i\sqrt{M}\langle\theta_M,X_{M^2}^{\Delta t}\rangle\big)\big]\\
&=\exp\Big(-M\sum_{m=\frac{M}{2}}^{M}\frac{1}{\lambda_m(2+\lambda_m\Delta t)}\big(1-\frac{1}{(1+\lambda_m\Delta t)^{2M^2}}\big)\Big)\\
&=\exp\Big(-\frac{1}{M}\sum_{m=\frac{M}{2}}^{M}\frac{1}{\pi^2(\frac{m}{M})^2(2+\pi^2(\frac{m}{M})^2)}+\mathrm{o}(1)\Big)\\
&\underset{M\to\infty}{\to}\exp\Big(-\int_{\frac12}^{1}\frac{1}{\pi^2z^2(2+\pi^2z^2)}dz\Big),
\end{aligned}
$$

using a Riemann sum argument, and $M\sum_{m=\frac{M}{2}}^{M}\frac{1}{\lambda_m(2+\frac{\lambda_m}{M^2})}\frac{1}{(1+\frac{\lambda_m}{M^2})^{2M^2}}=\mathrm{O}\big(\frac{1}{(1+\frac12)^{2M^2}}\big)\underset{M\to\infty}{\to}0$.

As a consequence, one obtains

$$
\limsup_{M\to\infty}\delta_2^2\Big(\frac{T}{M^2}\Big)\geq\exp\Big(-\int_{\frac12}^{1}\frac{1}{\pi^2z^2(2+\pi^2z^2)}dz\Big)-\exp\Big(-\int_{\frac12}^{1}\frac{1}{2\pi^2z^2}dz\Big)>0.
$$

Finally,

$$
\limsup_{N\to\infty}\delta_1^2(N)>0\ ,\quad\limsup_{\Delta t\to 0}\delta_2^2(\Delta t)>0,
$$

hence (11) is validated. This concludes the second proof of Theorem 1.

4.2. *Bounded Lipschitz test functions: proof of Theorem 2*

As already explained, it is sufficient to focus on the case $r\in(\frac14,\frac12)$.

The proof is based on introducing a family $\Phi^3=\big\{\phi_{\alpha,M}^3,\ \alpha\in(\frac14,\frac12],\ M\in\mathbb{N}\big\}\subset\mathcal{C}^{0,1}(H,\mathbb{R})$, of bounded and Lipschitz continuous test functions, such that $\|\phi\|_1\leq 1$ for all $\phi\in\Phi_3$. Precisely, for $\alpha\in(\frac14,\frac12]$ and $N\in\mathbb{N}$, set

$$
\phi_{\alpha,M}^3(x)=\frac{\exp\big(-\sum_{m=M}^{\infty}\frac{|\langle x,e_m\rangle|}{\lambda_m^{\alpha}}\big)}{1+\big(\sum_{m=1}^{\infty}\frac{1}{\lambda_m^{2\alpha}}\big)^{\frac12}}.
$$

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In contrast with the families Φ^1 and Φ^2 introduced above, note that functions in the set Φ^3 are not smooth. Introduce the notation $L_\alpha = 1 + \big(\sum_{m=1}^{\infty} \frac{1}{\lambda_m^{2\alpha}}\big)^{\frac{1}{2}} \in (0, \infty)$.

Let also

$$\delta_1^3(N) = \sup_{\phi\in\Phi_3} \Big|\int \phi d\mu_T - \int \phi d\mu_T^{(N)}\Big| \ , \quad \delta_2^3(\Delta t) = \sup_{\phi\in\Phi_3} \Big|\int \phi d\mu_T - \int \phi d\nu^{\Delta t}_{\lfloor \frac{T}{\Delta t}\rfloor}\Big|.$$

Let us introduce the following auxiliary function $f : [0, 1] \to \mathbb{R}$:

$$f(\theta) = -\log\big(\mathbb{E}[e^{-\theta|Z|}]\big),$$

where Z is a standard real-valued Gaussian random variable.

It is straightforward to check that f is of class $\mathcal{C}^\infty$ on $[0, 1]$, that it is bounded, and that all its derivatives are bounded. Moreover, $f(0) = 0$, and $f'(0) = \sqrt{\frac{2}{\pi}}$: it is crucial in the analysis below that $f'(0) \neq 0$.

4.2.1. Spatial discretization

First, focus on $\delta_1^3(N)$. For all $N \in \mathbb{N}$, and $\alpha \in (\frac{1}{4}, \frac{1}{2}]$, by the independence property of the components of the process X,

$$\mathbb{E}[\phi^3_{\alpha,1}(X(T))] = L_\alpha^{-1} \exp\Big(-\sum_{n=1}^{\infty} f\Big(\frac{\sigma_n(T)}{\sqrt{2\lambda_n}\lambda_n^\alpha}\Big)\Big),$$

$$\mathbb{E}[\phi^3_{\alpha,1}(X^{(N)}(T))] = L_\alpha^{-1} \exp\Big(-\sum_{n=1}^{N} f\Big(\frac{\sigma_n(T)}{\sqrt{2\lambda_n}\lambda_n^\alpha}\Big)\Big),$$

with $\sigma_n(T)^2 = 1 - e^{-2\lambda_n T}$. Thus

$$\begin{aligned}\mathbb{E}[\phi^3_{\alpha,1}(X(T))] - \mathbb{E}[\phi^3_{\alpha,1}(X^{(N)}(T))] &= \mathbb{E}[\phi^3_{\alpha,1}(X(T))]\Big(1 - \exp\Big(\sum_{n=N+1}^{\infty} f\Big(\frac{\sigma_n(T)}{\sqrt{2}\lambda_n^{\alpha+\frac{1}{2}}}\Big)\Big)\Big)\\ &= \mathbb{E}[\phi^3_{\alpha,1}(X(T))]\Big(1 - \exp\Big(\sum_{n=N+1}^{\infty} \Big(f'(0)\frac{\sigma_n(T)}{\sqrt{2}\lambda_n^{\alpha+\frac{1}{2}}} + \epsilon_n(T)\Big)\Big)\Big),\end{aligned}$$

where $\epsilon_n(T) = f\big(\frac{\sigma_n(T)}{\sqrt{2}\lambda_n^{\alpha+\frac{1}{2}}}\big) - f'(0)\frac{\sigma_n(T)}{\sqrt{2}\lambda_n^{\alpha+\frac{1}{2}}} = \mathrm{O}\big(\frac{\sigma_n(T)^2}{2\lambda_n^{2\alpha+1}}\big)$.

On the one hand, $\sum_{n=N+1}^{\infty} \epsilon_n(T) \underset{N\to\infty}{\to} 0$. On the other hand, when $N \to \infty$,

$$\begin{aligned}\sum_{n=N+1}^{\infty} \frac{\sigma_n(T)}{\sqrt{2}\lambda_n^{\alpha+\frac{1}{2}}} &= \sum_{n=N+1}^{\infty} \frac{1}{\sqrt{2}n^{2\alpha+1}\pi^{2\alpha+1}} + \mathrm{O}(e^{-\lambda_{N+1}T})\\ &\underset{N\to\infty}{\sim} \frac{C_\alpha}{N^{2\alpha}},\end{aligned}$$

with $C_\alpha = \int_1^\infty \frac{1}{\sqrt{2}\pi^{2\alpha+1}z^{2\alpha+1}} dz \in (0, \infty)$, by a Riemann sum argument.

Finally,

$$\mathbb{E}[\phi^3_{\alpha,1}(X(T))] - \mathbb{E}[\phi^3_{\alpha,1}(X^{(N)}(T))] \underset{N\to\infty}{\sim} \frac{f'(0)\mathbb{E}[\phi^3_{\alpha,1}(X(T))]C_\alpha}{\lambda_N^\alpha}.$$

We are now in position to conclude. Let $r \in (\frac{1}{4}, \frac{1}{2})$. Then, choosing $\alpha \in (\frac{1}{4}, r)$,

$$\limsup_{N\to\infty} \lambda_N^r \delta_1^3(N) \geq \limsup_{N\to\infty} \lambda_N^r \big|\mathbb{E}[\phi^3_{\alpha,1}(X(T))] - \mathbb{E}[\phi^3_{\alpha,1}(X^{(N)}(T))]\big| = \infty.$$

This concludes the proof of Theorem 2 for spatial discretization.

Remark 2. Note that, contrary to the other proofs in this article, a stronger result than Theorem 2 is obtained: let $\alpha \in (\frac{1}{4}, \frac{1}{2})$, then $\phi^3_{\alpha,1}$ is a test function for which the weak order of convergence is equal to α.

4.2.2. Temporal discretization

Now, focus on $\delta_2^3(\Delta t)$. It is convenient to choose $\Delta t = \frac{T}{M^2}$ and to consider functions $\phi^3_{\alpha,M}$. We claim that, for any $r \in (\frac{1}{4}, \frac{1}{2})$, choosing $\alpha \in (\frac{1}{4}, r)$, then

$$\limsup_{M\to\infty} M^{2r} \left| \mathbb{E}[\phi^3_{\alpha,M}(X(T))] - \mathbb{E}[\phi^3_{\alpha,M}(X^{\Delta t}_{M^2})] \right| = \infty. \tag{20}$$

On the one hand, the computations from the previous section prove that

$$\mathbb{E}[\phi^3_{\alpha,M}(X(T))] = L_\alpha^{-1} \exp\Big(- \sum_{m=M+1}^{\infty} f\big(\frac{\sigma_m(T)}{\sqrt{2}\lambda_m^{\alpha+\frac{1}{2}}}\big)\Big) = L_\alpha^{-1}\Big(1 - \frac{f'(0)C_\alpha}{M^{2\alpha}} + \mathrm{O}(\frac{1}{M^{4\alpha}})\Big).$$

On the other hand, using similar arguments (in particular, a Riemann sum appears),

$$\mathbb{E}[\phi^3_{\alpha,M}(X^{\Delta t}_{M^2})] = L_\alpha^{-1} \exp\Big(- \sum_{m=M+1}^{\infty} f\big(\frac{\sigma_m(T,M)}{\lambda_m^{\alpha+\frac{1}{2}}(2+\frac{\lambda_m}{M^2})^{\frac{1}{2}}}\big)\Big)$$
$$= L_\alpha^{-1}\Big(1 - \frac{f'(0)\overline{C}_\alpha}{M^{2\alpha}} + \mathrm{O}(\frac{1}{M^{4\alpha}})\Big),$$

with $\sigma_m(T,M)^2 = 1 - \frac{1}{(1+\frac{\lambda_m}{M^2})^{2M^2}}$, and

$$\overline{C}_\alpha = \int_1^\infty \frac{1}{\sqrt{2+\pi^2 z^2}\pi^{2\alpha+1} z^{2\alpha+1}} dz < C_\alpha.$$

Thus

$$\mathbb{E}[\phi^3_{\alpha,M}(X(T))] - \mathbb{E}[\phi^3_{\alpha,M}(X^{\Delta t}_{M^2})] \underset{M\to\infty}{\sim} \frac{f'(0)(\overline{C}_\alpha - C_\alpha)}{L_\alpha M^{2\alpha}}.$$

This expression implies the claim (20) holds true, hence

$$\limsup_{\Delta t\to 0} \frac{1}{\Delta t^r}\delta_2^3(\Delta t) = \infty.$$

This concludes the proof of Theorem 2 for temporal discretization.

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