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Strong Rates of Convergence of a Splitting Scheme for Schrödinger Equations with Nonlocal Interaction Cubic Nonlinearity and White Noise Dispersion*

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Abstract. We analyze a splitting integrator for the time discretization of the Schrödinger equation with nonlocal interaction cubic nonlinearity and white noise dispersion. We prove that this time integrator has order of convergence one in the pth mean sense, for any $p \geq 1$ in some Sobolev spaces. We prove that the splitting schemes preserves the L^2-norm, which is a crucial property for the proof of the strong convergence result. Finally, numerical experiments illustrate the performance of the proposed numerical scheme.

Key words. stochastic Schrödinger equations, white noise dispersion, nonlocal interaction cubic nonlinearity, splitting integrators, strong convergence rates

AMS subject classifications. 65C30, 65J08, 60H15, 60H35, 60-08, 35Q55

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1. Introduction.

We consider the time discretization by a splitting scheme for the following class of nonlinear Schrödinger equations with white noise dispersion:

$$\begin{cases} \mathrm{i}\mathrm{d}u(t) + \Delta u(t) \circ \mathrm{d}\beta(t) + V[u(t)]u(t)\,\mathrm{d}t = 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where the unknown $u = u(t, \cdot)$, with $t \geq 0$, is a complex valued random process defined on $\mathbb{R}^d$, $\Delta u = \sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2}$ denotes the Laplacian in $\mathbb{R}^d$, and $\beta = \beta(t)$ is a real-valued standard Brownian motion. The nonlinearity $\Psi_0(u) = V[u]u$ in the stochastic partial differential equation (SPDE) (1.1) is a nonlocal interaction cubic nonlinearity, $V[u] = V * |u|^2 = \int V(\cdot - x)|u(x)|^2\,\mathrm{d}x$, where $*$ denotes the convolution operator and the real-valued mapping $V\colon \mathbb{R}^d \to \mathbb{R}$ is at least continuous and bounded; more precise regularity conditions are imposed below. Such long-range interaction is a smooth version of the nonlinearity in the (deterministic) Hartree or Schrödinger–Poisson equations (see, for instance, [14]). Splitting schemes for Schrödinger equations driven by additive space-time noise with this type of nonlinearity were recently

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studied in [3]. Observe that the case of power-law nonlinearities cannot be treated by the techniques employed in the present publication. The SPDE (1.1) is understood in the Stratonovich sense, using the $\circ$ symbol for the Stratonovich product.

Theoretical results on well-posedness of the SPDE (1.1) are relatively scarce and given mostly for the case of a power-law nonlinearity ($|u|^{2\sigma}u$ for σ a positive real number) in place of the nonlocal interaction nonlinearities considered in this article. For instance, it has been shown that SPDEs of the type (1.1) with power-law nonlinearities have solutions in H^1 for dimension $d = 1$ and $\sigma = 2$ [8, Theorem 2.2] and for $\sigma < 2/d$ in any dimension [9, Theorem 2.3].

To the best of our knowledge, no strong convergence rates are know for a time discretization of the SPDE (1.1) with the considered type of locally Lipschitz nonlinearity. However, strong convergence results have been proved in the case of a globally Lipschitz nonlinearity in place of the above nonlinearity. In addition, rates of convergence in probability for a pure cubic nonlinearity in place of the above nonlocal interaction cubic nonlinearity have also been obtained. We now review these known convergence results. The work [15] studies a Lie–Trotter splitting integrator. The mean-square order of convergence of this explicit numerical method is proven to be at least 1/2 for a (truncated) Lipschitz nonlinearity [15, sections 5 and 6]. Furthermore, [15] conjectures that this splitting scheme should have strong order one and supports this conjecture numerically. Sharp order estimates for the same splitting scheme (but applied to a more general problem) were recently presented in the preprint [16]. The authors of [1] study a semi-implicit Crank–Nicolson scheme. In particular, they show that this time integrator has mean-square order of convergence one for a truncated problem and order of convergence in probability one in the case of a cubic nonlinearity. For the same problem, the same convergence rates are obtained for a multisymplectic integrator in [7] and for an explicit exponential scheme in [6]. We conclude this list of references with the recent work [13], which considers a randomized exponential integrator for time discretization of related (nonrandom) nonlinear modulated Schrödinger equations.

In the present publication, we consider an explicit splitting integrator for an efficient time discretization of the nonlinear stochastic Schrödinger equation (1.1). In essence, the main principle of splitting integrators is to decompose the vector field of the original differential equation in several parts, such that the arising subsystems are exactly (or easily) integrated. We refer interested readers to [12, 2, 17] for details on splitting schemes for ordinary and partial differential equations. The splitting scheme considered in this publication is given by

$$u_{n+1} = \mathrm{e}^{\mathrm{i}(\beta(t_{n+1})-\beta(t_n))\Delta}\big(\mathrm{e}^{\mathrm{i}\tau V[u_n]}u_n\big),$$

where τ denotes the time-step size, $t_n = n\tau$, and $u_n \approx u(t_n)$ (see (4.2) below for details). In the scheme above, $\mathrm{e}^{\mathrm{i}\tau V[u_n]}$ acts as a pointwise multiplication operator.

The main result of this paper is a strong convergence result for the explicit and easy to implement splitting integrator for the time discretization of (1.1) defined above (see section 4 for a precise statement). Theorem 4.5 states that the splitting scheme converges with order 1 in the $L^p(\Omega, H^m(\mathbb{R}^d, \mathbb{C}))$ sense for all $p \in [1, \infty)$ (whereas only $p = 2$ is treated in related works with truncated globally Lipschitz continuous nonlinearities). We also obtain convergence with order 1 in probability and in the almost sure sense. A crucial property for showing these results is the fact that the splitting scheme exactly preserves the L^2-norm as does the exact solution to (1.1) (see Propositions 3.1 and 4.1).

Since the considered nonlinearity is not globally Lipschitz continuous, and since the proposed scheme is explicit, we need to face several challenges in the strong error analysis. For instance, we obtain moment bounds in H^m norms for the numerical solution, which follow from the preservation of the L^2-norm by the proposed integrator. This property is not satisfied for the explicit exponential scheme from [6]. In addition, even if the L^2-norm is preserved for the implicit schemes studied in [1, 7], in these two references, strong error estimates are obtained only for truncated (globally Lipschitz continuous) nonlinearities. In the present article, the nonlocal interaction nonlinearity $u \mapsto V[u]u$ is locally Lipschitz continuous from L^2 to L^2 (and even from H^m to H^m; see Lemma 2.2). Note that if one would replace the nonlinearity in the SPDE (1.1) by a pure cubic nonlinearity $u \mapsto |u|^2u$ (which is not locally Lipschitz continuous from L^2 to L^2) as in the references [1, 6, 7], the proposed splitting integrator would still preserve the L^2-norm but only convergence in probability would be obtained. However, their authors do not establish the appropriate moment bounds in suitable H^m norms; this explains why they cannot prove strong convergence results. For the considered nonlinearity, we are able to establish the required moment bounds and to give a strong convergence result with rate 1. It is worth mentioning that some error terms need to be controlled very carefully to obtain order 1 instead of order 1/2, which is the expected strong order of convergence for temporal discretization of stochastic problems (see Remark 5.2). Thus our main result and its proof are not straightforward extensions of existing results.

To the best of our knowledge, Theorem 4.5 is the first strong convergence result obtained for a time discretization scheme applied to the nonlinear Schrödinger equation with white noise dispersion with a nonglobally Lipschitz continuous nonlinearity.

In order to show such convergence results, we begin the exposition by introducing some notation and recalling useful results in section 2. Section 3 then provides various properties of the exact solution to the SPDE (1.1). After that, we present the splitting scheme and its main properties (preservation of the L^2-norm, bounds for the H^m norms, symplecticity, asymptotic preserving property) and analyze its strong convergence in section 4. The proof of the main convergence result is given in section 5. Several numerical experiments in dimensions 1 and 2 illustrating the main properties of the proposed numerical scheme are presented in section 6.

Throughout this article, we denote by C a generic constant that may vary from line to line. Furthermore, we set $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, \ldots\}$. Finally, the initial value u_0 of the SPDE (1.1) is assumed to be nonrandom for ease of presentation. The results of this paper can be extended to the case of random u_0 (independent of the given Brownian motion and with appropriate moment bounds).

2. Setting and useful results. We denote the classical Lebesgue space of complex functions by $L^2 = L^2(\mathbb{R}^d, \mathbb{C})$, endowed with its real vector space structure, and with the inner product

$$(u, v) = \operatorname{Re} \int_{\mathbb{R}^d} u\bar{v} \, \mathrm{d}x = \operatorname{Re} \int u\bar{v} \, \mathrm{d}x$$

as well as its norm denoted by $\|\cdot\|_{L^2}$. For $m \in \mathbb{N}$, we further denote by $H^m = H^m(\mathbb{R}^d, \mathbb{C})$ the Sobolev space of functions in L^2, with weak derivatives of order $1, \ldots, m$ in L^2. Let also

$H^0 = L^2$. The Fourier transform of a tempered distribution v is denoted by $\widehat{v}$. With this notation, H^m is the Sobolev space of tempered distributions v such that $(1+|\zeta|^2)^{m/2}\widehat{v} \in L^2$. The Sobolev space H^m is equipped with the norm defined by

$$\|v\|_{H^m}^2 = \sum_{|\alpha|\le m} \|\partial_\alpha v\|_{L^2}^2 ,$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ is a multi-index and $|\alpha| = \sum_{i=1}^d \alpha_i$. Note that if $m_1 \le m_2$, one has $H^{m_2} \subset H^{m_1}$ and $\|v\|_{H^{m_1}} \le \|v\|_{H^{m_2}}$ for all $v \in H^{m_2}$. If α and γ are two multi-indices, it is said that $\gamma \le \alpha$ if $\gamma_i \le \alpha_i$ for all $i = 1, \dots, d$. If $\gamma \le \alpha$, we also introduce the notation $\alpha - \gamma = (\alpha_i - \gamma_i)_{1\le i\le d}$ and $\binom{\alpha}{\gamma} = \prod_{i=1}^d \binom{\alpha_i}{\gamma_i}$.

The Banach space $\mathcal{C}^m = \mathcal{C}_b^m(\mathbb{R}^d, \mathbb{C})$ of complex-valued continuous and bounded functions that have continuous and bounded derivatives of order $1, \dots, m$ is equipped with the norm

$$\|v\|_{\mathcal{C}^m} = \sup_{|\alpha|\le m} \sup_{x\in\mathbb{R}^d} |\partial_\alpha v(x)|.$$

Let us recall a version of the Leibniz rule. See, for instance, [11, section 5.2.3, Theorem 1] for a proof in the case of smooth compactly supported functions $v \in \mathcal{C}_c^\infty$ and [5, Theorem 8.25] for the argument to extend the result to $v \in \mathcal{C}^m$.

Lemma 2.1 (Leibniz rule). *For all $m \in \mathbb{N}$, there exists $C_m \in (0, \infty)$ such that for all $u \in H^m$ and $v \in \mathcal{C}^m$, one has $uv \in H^m$, and*

$$\|uv\|_{H^m} \le C_m \|u\|_{H^m} \|v\|_{\mathcal{C}^m} .$$

In addition, the Leibniz rule holds: for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$, one has

$$\partial_\alpha(uv) = \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \partial_\gamma u \partial_{\alpha-\gamma} v.$$

Let $\beta = \big(\beta(t)\big)_{t\ge 0}$ be a standard real-valued Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0})$ satisfying the usual conditions.

For all $t, s \ge 0$, define the operator $S(t,s)$ as follows (see, for instance, [8, section 3] and [15, section 3]):

$$S(t,s) = \mathrm{e}^{\mathrm{i}(\beta(t)-\beta(s))\Delta}. \tag{2.1}$$

Note that $S(t,s) = \mathcal{S}(\beta(t) - \beta(s))$, where $\mathcal{S}(r) = e^{\mathrm{i}r\Delta}$ defines the standard group associated with the linear Schrödinger equation $\mathrm{i}\mathrm{d}u(r) + \Delta u(r)\mathrm{d}r = 0$. In Fourier variables, one has the expression

$$\widehat{S(t,s)v}(\zeta) = \exp\Big(-\mathrm{i}\,|\zeta|^2\,(\beta(t) - \beta(s))\Big)\,\widehat{v}(\zeta)$$

for all $t, s \ge 0$, all $\zeta \in \mathbb{R}^d$, and any $v \in H^m$, $m \in \mathbb{N}_0$.

The operators $S(t,s)$ for $t \ge s$ play an important role in this work: if v is an $\mathcal{F}_s$-measurable random function with values in H^m, then $t \mapsto v_s(t) = S(t,s)v$ is the solution of the stochastic linear Schrödinger equation

$$\mathrm{i}\mathrm{d}v_s(t) + \Delta v_s(t) \circ \mathrm{d}\beta(t) = 0, \quad t \ge s,$$

with $v_s(s) = v$.

Two properties of the operators $S(t,s)$ will be used repeatedly in this article. First, for all $m \in \mathbb{N}_0$, all $t, s \geq 0$, and all $v \in H^m$, one has the isometry property

$$\|S(t,s)v\|_{H^m} = \|v\|_{H^m}\,. \tag{2.2}$$

Second, for all $m \in \mathbb{N}_0$, all $t, s \geq 0$, and all $v \in H^{m+2}$, one has

$$\|S(t,s)v - v\|_{H^m} \leq |\beta(t) - \beta(s)|\,\|v\|_{H^{m+2}}\,. \tag{2.3}$$

Let us now study properties of the nonlinearity in the SPDE (1.1) defined by

$$\Psi_0(u) = V[u]u = \big(V * |u|^2\big)u.$$

If $V \in \mathcal{C}^m$, the mapping $\Psi_0 : H^m \to H^m$ is well-defined and is locally Lipschitz continuous. More precisely, one has the following result.

Lemma 2.2. *Let $m \in \mathbb{N}_0$ and assume that $V \in \mathcal{C}^m$. There exists $C_m(V) \in (0,\infty)$ such that the following properties hold.*

First, for all $u \in H^m$, one has $\Psi_0(u) \in H^m$ and

$$\|\Psi_0(u)\|_{H^m} \leq C_m(V)\|u\|_{L^2}^2\|u\|_{H^m}. \tag{2.4}$$

In addition, Ψ_0 is locally Lipschitz continuous in H^m: for all $u_1, u_2 \in H^m$, one has

$$\|\Psi_0(u_2) - \Psi_0(u_1)\|_{H^m} \leq C_m(V)\left(\|u_1\|_{H^m}^2 + \|u_2\|_{H^m}^2\right)\|u_2 - u_1\|_{H^m}\,. \tag{2.5}$$

Finally, Ψ_0 is twice differentiable, and its first and second order derivatives satisfy the following result: for all $u, h, k \in H^m$, one has

$$\left\|\Psi_0'(u).h\right\|_{H^m} \leq C_m(V)\,\|u\|_{L^2}\,\|u\|_{H^m}\,\|h\|_{H^m} \tag{2.6}$$

and

$$\left\|\Psi_0''(u).(h,k)\right\|_{H^m} \leq C_m(V)\,\|u\|_{L^2}\,\|h\|_{H^m}\,\|k\|_{H^m}\,. \tag{2.7}$$

Proof of Lemma 2.2. Let $m \in \mathbb{N}_0$ be fixed.

Using the definition of Ψ_0, the Leibniz rule (Lemma 2.1), and the property $\|V[u]\|_{\mathcal{C}^m} = \left\|V * |u|^2\right\|_{\mathcal{C}^m} \leq \|V\|_{\mathcal{C}^m}\,\|u\|_{L^2}^2$, the proof of (2.4) is straightforward: for all $u \in H^m$, one has

$$\|\Psi_0(u)\|_{H^m} = \|V[u]u\|_{H^m} \leq C_m\,\|V[u]\|_{\mathcal{C}^m}\,\|u\|_{H^m} \leq C_m\,\|V\|_{\mathcal{C}^m}\,\|u\|_{L^2}^2\,\|u\|_{H^m}\,.$$

To prove (2.6) and (2.7), note that the expressions for the derivatives are given by

$$\begin{aligned}\Psi_0'(u).h &= V[u]h + 2V * (\mathrm{Re}(\bar{u}h))\,u,\\ \Psi_0''(u).(h,k) &= 4V * \big(\mathrm{Re}(\bar{k}h)\big)\,u + 2V * (\mathrm{Re}(\bar{u}k))\,h + 2V * (\mathrm{Re}(\bar{u}h))\,k.\end{aligned}$$

Using the Leibniz rule (Lemma 2.1) again, one obtains

$$\begin{aligned}\left\|\Psi_0'(u).h\right\|_{H^m} &\le \|V[u]\|_{\mathcal{C}^m}\|h\|_{H^m} + 2\|V*(\mathrm{Re}(\bar{u}h))\|_{\mathcal{C}^m}\|u\|_{H^m}\\ &\le \|V\|_{\mathcal{C}^m}\|u\|_{L^2}^2\|h\|_{H^m} + 2\|V\|_{\mathcal{C}^m}\|u\|_{L^2}\|h\|_{L^2}\|u\|_{H^m}\end{aligned}$$

and

$$\begin{aligned}&\left\|\Psi_0''(u).(h,k)\right\|_{H^m}\\ &\le 4\left\|V*\big(\mathrm{Re}(\bar{k}h)\big)\right\|_{\mathcal{C}^m}\|u\|_{H^m}\\ &+2\|V*(\mathrm{Re}(\bar{u}k))\|_{\mathcal{C}^m}\|h\|_{H^m} + 2\|V*(\mathrm{Re}(\bar{u}h))\|_{\mathcal{C}^m}\|k\|_{H^m}\\ &\le 4\|V\|_{\mathcal{C}^m}\left(\|u\|_{H^m}\|h\|_{L^2}\|k\|_{L^2} + \|u\|_{L^2}\|h\|_{H^m}\|k\|_{L^2} + \|u\|_{L^2}\|h\|_{L^2}\|k\|_{H^m}\right).\end{aligned}$$

Finally, in order to prove (2.5), it suffices to write

$$\Psi_0(u_2)-\Psi_0(u_1)=\int_0^1 \Psi_0'\big((1-\xi)u_1+\xi u_2\big).(u_2-u_1)\,\mathrm{d}\xi$$

and to use (2.6). One then obtains

$$\|\Psi_0(u_2)-\Psi_0(u_1)\|_{H^m} \le C_m(V)(\|u_1\|_{H^m}^2+\|u_2\|_{H^m}^2)\,\|u_2-u_1\|_{H^m}.$$

This concludes the proof of Lemma 2.2. ■

3. Properties of the exact solution. In this section, we provide a well-posedness result and some properties of the exact solution $u(t)$ of the nonlinear Schrödinger equation with white noise dispersion (1.1).

Proposition 3.1. *Assume that $V \in \mathcal{C}^0$.*

For any (nonrandom) initial condition $u_0 \in L^2$, there exists a unique mild solution $(u(t))_{t\ge 0}$ of the Schrödinger with white noise dispersion (1.1) in L^2, which means that for all $t \ge 0$ one has

$$u(t) = S(t,0)u_0 + \mathrm{i}\int_0^t S(t,r)\,(V[u(r)]u(r))\,\mathrm{d}r, \tag{3.1}$$

where $\big(S(t,s)\big)_{t\ge s\ge 0}$ is defined by (2.1).

In addition, one has conservation of the L^2-norm: for all $t \ge 0$, one has almost surely

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}. \tag{3.2}$$

Furthermore, the SPDE (1.1) is a stochastic Hamiltonian system, in the sense of [7, section 2]*, and thus its solution preserves the stochastic symplectic structure*

$$\bar{\omega} = \int_{\mathbb{R}^d} \mathrm{d}p\wedge \mathrm{d}q\,\mathrm{d}x \quad \textit{almost surely},$$

where the overbar on ω is a reminder that the two-form $\mathrm{d}p\wedge\mathrm{d}q$ (with differentials made with respect to the initial value) is integrated over $\mathbb{R}^d$. Here, $p(t)=\mathrm{Re}(u(t))$ and $q(t)=\mathrm{Im}(u(t))$ denote the real and imaginary parts of $u(t)$.

Moreover, one can bound the solution in H^m in the following sense. Let $m \in \mathbb{N}$ and assume that $V \in \mathcal{C}^m$. There exists $C_m(V) \in (0,\infty)$ such that if $u_0 \in H^m$, then almost surely $u(t) \in H^m$ for all $t \geq 0$, and

$$\|u(t)\|_{H^m} \leq \mathrm{e}^{C_m(V)\|u_0\|_{L^2}^2 t} \|u_0\|_{H^m} . \tag{3.3}$$

Finally, for all $u_0 \in H^{m+2}$, $T \in (0,\infty)$, and $p \in [1,\infty)$, there exists $C_p(T, \|u_0\|_{H^{m+2}}) \in (0,\infty)$ such that for all $0 \leq t_1 \leq t_2 \leq T$, one has

$$\left(\mathbb{E}[\|u(t_2) - u(t_1)\|_{H^m}^p]\right)^{\frac{1}{p}} \leq C_p(T, \|u_0\|_{H^{m+2}})(t_2 - t_1)^{\frac{1}{2}}. \tag{3.4}$$

Proof. Since Ψ_0 is locally Lipschitz continuous from H^m to H^m, if $V \in \mathcal{C}^m$, local well-posedness of mild solutions in H^m is a standard result.

To prove that solutions to (1.1) are global, we use a truncation argument. Let $\theta \colon [0,\infty) \to [0,1]$ be a compactly supported Lipschitz continuous function, such that $\theta(x) = 1$ for $x \in [0,1]$. For any $R \in (0,\infty)$, set $V^R(u) = \theta(R^{-1}\|u\|_{L^2})V[u]$ and $F^R(u) = V^R(u)u$. The mapping F^R is globally Lipschitz continuous, and the SPDE

$$\mathrm{i}\mathrm{d}u^R(t) + \Delta u^R(t) \circ \mathrm{d}\beta(t) + F^R(u^R(t))\,\mathrm{d}t = 0$$

with initial condition $u^R(0) = u_0$, admits a unique global mild solution $\left(u^R(t)\right)_{t\in[0,T]}$, where T is an arbitrary positive real number. Applying Itô's formula to a regularization of $u^R(t)$ as in the proof of [8, Theorem 4.1], for instance, one checks that $\left\|u^R(t)\right\|_{L^2} = \left\|u^R(0)\right\|_{L^2}$ for all $t \in [0,T]$. Choosing $R > \|u_0\|_{L^2}$ shows that one can define $u(t) = u^R(t)$ for all $t \geq 0$. Then $u(t)$ is the unique solution on $[0,T]$ of the fixed point equation (3.1), i.e., $u(t)$ is the unique mild solution of (1.1), and one has the preservation of the L^2-norm (3.2).

The fact that the problem (1.1) is a stochastic Hamiltonian system is seen, exactly as in [7, section 2], by considering its real and imaginary parts and observing that the obtained differential equations are indeed stochastic Hamiltonian systems. The preservation of the stochastic symplectic structure follows also as in [7, section 2] since the potential V in (1.1) is real-valued (as opposed to a power-law nonlinearity $|u|^{2\sigma}$ in the above reference).

Let us now prove the bound in H^m (see (3.3)). Using Lemma 2.2, one obtains

$$\|V[u]u\|_{H^m} \leq C\|V[u]\|_{\mathcal{C}^m}\|u\|_{H^m} \leq C\|V\|_{\mathcal{C}^m}\|u\|_{L^2}^2\|u\|_{H^m} .$$

Then, using the isometry property for $S(t,s)$ in H^m (see (2.2)), the mild formulation (3.1), and the preservation of the L^2-norm (3.2), one then obtains

$$\begin{aligned}\|u(t)\|_{H^m} &\leq \|u_0\|_{H^m} + \int_0^t \|V[u(s)]u(s)\|_{H^m}\,\mathrm{d}s\\ &\leq \|u_0\|_{H^m} + C\int_0^t \|u(s)\|_{L^2}^2\|u(s)\|_{H^m}\,\mathrm{d}s\\ &\leq \|u_0\|_{H^m} + C\int_0^t \|u_0\|_{L^2}^2\|u(s)\|_{H^m}\,\mathrm{d}s.\end{aligned}$$

Applying Gronwall's lemma then yields (3.3).

It remains to establish the temporal regularity property (3.4). Using the mild formulation (3.1) and the isometry property (2.2), one obtains

$$\begin{aligned}\|u(t_2)-u(t_1)\|_{H^m} &\le \|(S(t_2,t_1)-I)\,u(t_1)\|_{H^m} + \int_{t_1}^{t_2} \|V[u(t)]u(t)\|_{H^m}\,\mathrm{d}t \\ &\le \|(S(t_2,t_1)-I)\,u(t_1)\|_{H^m} + C\,\|u_0\|_{L^2}^2 \int_{t_1}^{t_2} \|u(t)\|_{H^m}\,\mathrm{d}t,\end{aligned}$$

using the inequality $\|V[u(t)]u(t)\|_{H^m} \le C\,\|u(t)\|_{L^2}^2\,\|u(t)\|_{H^m} \le C\,\|u_0\|_{L^2}^2\,\|u(t)\|_{H^m}$, owing to the preservation of the L^2-norm (3.2).

Finally, using (2.3), the fact that $\left(\mathbb{E}[|\beta(t_2)-\beta(t_1)|^p]\right)^{\frac{1}{p}} \le C_p|t_2-t_1|^{\frac{1}{2}}$, and the bound for the exact solution in the H^m norm (3.3), one obtains, for all $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned}(\mathbb{E}[\|u(t_2)-u(t_1)\|_{H^m}^p])^{\frac{1}{p}} &\le C_p(t_2-t_1)^{1/2}\mathrm{e}^{C_{m+2}(V)\|u_0\|_{L^2}^2 t_1}\,\|u_0\|_{H^{m+2}} \\ &\quad + C(t_2-t_1)\mathrm{e}^{C_m(V)\|u_0\|_{L^2}^2 t_2}\,\|u_0\|_{H^m}^3 \\ &\le C_p(T,\|u_0\|_{H^{m+2}})(t_2-t_1)^{1/2}.\end{aligned}$$

This yields (3.4) and concludes the proof of Proposition 3.1. ∎

4. Numerical analysis of the splitting scheme. In this section, we propose and study an efficient time integrator for the SPDE (1.1). We state and prove some properties of the numerical solution, in particular preservation of the L^2-norm (Proposition 4.1). Furthermore, we state the main strong convergence result (Theorem 4.5) of the paper, namely, that the splitting scheme has convergence rate 1. Finally, we deduce various auxiliary results from the main theorem.

4.1. Presentation of the splitting scheme. Let $T > 0$ be a fixed time horizon and an integer $N \ge 1$. We define the step size of the numerical method by $\tau = T/N$ and denote the discrete times by $t_n = n\tau$ for $n = 0,\dots,N$. Without loss of generality, we assume that $\tau \in (0,1)$.

The main idea of a splitting integrator for the SPDE (1.1) is based on the observation that the vector field of the original problem can be decomposed into two parts (linear and nonlinear parts, resp.) that are exactly integrated.

On the one hand, the solution of the linear stochastic evolution equation

$$\mathrm{i}\mathrm{d}u(t) + \Delta u(t) \circ \mathrm{d}\beta(t) = 0, \quad u(0) = u_0,$$

is given by $u(t) = S(t,0)u_0$, where the random propagator $S(t,0)$ is defined by (2.1).

On the other hand, the solution of the nonlinear evolution equation

$$\mathrm{i}\mathrm{d}u(t) + V[u(t)]u(t)\,\mathrm{d}t = 0, \quad u(0) = u_0,$$

is given by $u(t) = \Phi_t(u_0)$, where for all $t \ge 0$ and all $u \in L^2$, one has

$$\Phi_t(u) = \mathrm{e}^{\mathrm{i}tV[u]}u. \tag{4.1}$$

The Lie–Trotter splitting strategy yields the definition of the following time integrator for the nonlinear Schrödinger equation with white noise dispersion (1.1):

$$u_{n+1} = S(t_{n+1}, t_n)\Phi_\tau(u_n). \tag{4.2}$$

The following notation will be used in what follows: for all $\tau \in (0,1)$ and $u \in L^2$, set

$$\Psi_\tau(u) = \frac{\Phi_\tau(u) - u}{\mathrm{i}\tau}.$$

4.2. Properties of the numerical solution. This subsection lists useful properties of the numerical solution given by the splitting scheme (4.2).

Conservation of the L^2-norm. The splitting scheme exactly preserves the L^2-norm as does the exact solution to the SPDE (1.1) (see (3.2) in Proposition 3.1). This conservation property plays a crucial role in the error analysis presented below.

Proposition 4.1. *Let $u_0 \in L^2$, $\tau \in (0,1)$ and let $\big(u_n\big)_{n\in\mathbb{N}_0}$ be given by the splitting scheme (4.2), one then has conservation of the L^2-norm: for all $n \in \mathbb{N}$, one has almost surely*

$$\|u_n\|_{L^2} = \|u_0\|_{L^2}. \tag{4.3}$$

Proof. Using the isometry property (2.2), then the definition (4.1) of the flow Φ_τ, a direct computation from the definition of the scheme (4.2) gives for all $n \in \mathbb{N}_0$

$$\|u_{n+1}\|_{L^2} = \|S(t_{n+1}, t_n)\Phi_\tau(u_n)\|_{L^2} = \|\Phi_\tau(u_n)\|_{L^2} = \|u_n\|_{L^2}.$$

A straightforward recursion argument concludes the proof. ∎

Bounds for the numerical solution in $\mathbf{H^m}$.

Proposition 4.2 below states almost sure upper bounds for the numerical solution $\|u_n\|_{H^m}$ for all $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$.

Proposition 4.2. *Let $m \in \mathbb{N}$ and assume that $V \in \mathcal{C}^m$. There exists $C_m(V) \in (0,\infty)$, such that for any initial condition $u_0 \in H^m$, the numerical solution u_n defined by the splitting scheme (4.2) satisfies the following upper bound: for all $n \in \mathbb{N}_0$, one has almost surely*

$$\|u_n\|_{H^m} \le \mathrm{e}^{C_m(V)t_n\|u_0\|_{L^2}^{2m}} \|u_0\|_{H^m}. \tag{4.4}$$

The proof of Proposition 4.2 requires the following auxiliary result.

Lemma 4.3. *Let $m \in \mathbb{N}$ and assume that $V \in \mathcal{C}^m$. There exists $C_m(V) \in (0,\infty)$ such that for all $\tau \in (0,1)$ and all $u \in H^m$, one has*

$$\|\Phi_\tau(u)\|_{H^m} \le \Big(1 + C_m(V)\tau(1 + \|u\|_{L^2}^{2m})\Big) \|u\|_{H^m}.$$

Proof of Lemma 4.3. Using the definition (4.1), one has the identity $\Phi_\tau(u) = \mathrm{e}^{\mathrm{i}\tau V[u]}u$ with $V[u] = V * |u|^2 \in \mathcal{C}^m$ and $\mathrm{e}^{\mathrm{i}\tau V[u]} \in \mathcal{C}^m$, since $V \in \mathcal{C}^m$.

The following expression holds: for all $u \in H^m$ one has

$$\Phi_\tau(u) - u = \theta_\tau(V[u])u,$$

where $\theta_\tau(y) = \mathrm{e}^{\mathrm{i}\tau y} - 1$ for all $y \in \mathbb{R}$. Applying the inequality from Lemma 2.1, one has, for all $u \in H^m$,

$$\|\Phi_\tau(u) - u\|_{H^m} \leq C_m \|\theta_\tau(V[u])\|_{\mathcal{C}^m} \|u\|_{H^m} .$$

It remains to study the behavior of $\|\theta_\tau(V[u])\|_{\mathcal{C}^m}$. The auxiliary function θ_τ satisfies the following properties: for all $y \in \mathbb{R}$, all $\tau \in (0,1)$, and all $k \in \mathbb{N}$,

$$|\theta_\tau(y)| \leq \tau |y|,$$
$$|\theta_\tau^{(k)}(y)| \leq \tau^k.$$

Using the Faà di Bruno formula, one obtains the bounds

$$\|\partial_\gamma \theta_\tau(V[u])\|_{\mathcal{C}^0} \leq C \begin{cases} \tau \|V[u]\|_{\mathcal{C}^0} \ , \quad \gamma = 0, \\ C_{|\gamma|}\tau \left(1 + \|V[u]\|_{\mathcal{C}^{|\gamma|}}^{|\gamma|}\right) \ , \quad \gamma \neq 0. \end{cases}$$

Finally, using the inequality

$$\|V[u]\|_{\mathcal{C}^{|\gamma|}}^{|\gamma|} = \left\|\partial_\gamma \left(V * |u|^2\right)\right\|_{\mathcal{C}^0}^{|\gamma|} \leq C_m(V) \|u\|_{L^2}^{2m} ,$$

if $1 \leq |\gamma| \leq m$, then the proof of Lemma 4.3 concludes. ■

We are now in position to provide the proof of Proposition 4.2.

Proof of Proposition 4.2. Using the definition (4.2) of the splitting scheme, the isometry property (2.2) of the random propagator $S(t_{n+1}, t_n)$, and Lemma 4.3, one gets

$$\begin{aligned} \|u_{n+1}\|_{H^m} &= \|S(t_{n+1}, t_n)\Phi_\tau(u_n)\|_{H^m} = \|\Phi_\tau(u_n)\|_{H^m} \\ &\leq \left(1 + C_m(V) \|u_n\|_{L^2}^{2m} \tau\right) \|u_n\|_{H^m} . \end{aligned}$$

Using the preservation property (4.3) of the L^2-norm by the splitting integrator (see Proposition 4.1), one then obtains the following estimate: for all $n \in \mathbb{N}_0$,

$$\|u_{n+1}\|_{H^m} \leq \left(1 + C_m(V) \|u_0\|_{L^2}^{2m} \tau\right) \|u_n\|_{H^m} .$$

Finally, a straightforward recursion argument yields the following bound: for all $n \in \mathbb{N}_0$, one has

$$\|u_n\|_{H^m} \leq \mathrm{e}^{C_m(V) t_n \|u_0\|_{L^2}^{2m}} \|u_0\|_{H^m} .$$

All the estimates above hold in an almost sure sense. This concludes the proof of Proposition 4.2. ■

Numerical preservation of the stochastic symplectic structure. As seen in Proposition 3.1, the exact solution to the SPDE (1.1) preserves a stochastic symplectic structure. The next result states that the same geometric structure is also preserved by the splitting scheme (4.2).

Proposition 4.4. *Consider the numerical discretization of the Schrödinger equation with white noise dispersion* (1.1) *by the splitting scheme* (4.2). *Then, the splitting scheme preserves the stochastic symplectic structure*

$$\bar{\omega}^{n+1} = \bar{\omega}^n \quad \text{for} \quad n = 0, \ldots, N-1 \quad \text{almost surely,}$$

where $\bar{\omega}^n = \int_{\mathbb{R}^d} \mathrm{d}p^n \wedge \mathrm{d}q^n \,\mathrm{d}x$ *and* p^n, *resp.*, q^n, *are the real, resp., imaginary parts of* u^n.

Proof. The splitting integrator (4.2) is obtained by solving exactly sequentially the following differential equations:

$$\mathrm{i}\mathrm{d}u(t) + V[u(t)]u(t)\,\mathrm{d}t = 0$$

and

$$\mathrm{i}\mathrm{d}u(t) + \Delta u(t) \circ \mathrm{d}\beta(t) = 0.$$

Considering the real and imaginary parts of these differential equations and using the fact that V is real-valued, one gets

$$\mathrm{d}p(t) = -V[(p(t), q(t))]q(t)\,\mathrm{d}t, \quad \mathrm{d}q(t) = V[(p(t), q(t))]p(t)\,\mathrm{d}t$$

and

$$\mathrm{d}p(t) = -\Delta q(t) \circ \mathrm{d}\beta(t), \quad \mathrm{d}q(t) = \Delta p(t) \circ \mathrm{d}\beta(t).$$

The above problems are infinite-dimensional stochastic Hamiltonian systems in the sense of [7, equation (6)]. It thus follows, as in [7, Proposition 3.3], that the splitting scheme preserves the stochastic symplectic structures of each of these Hamiltonian systems, as it is obtained as composition of symplectic maps, and hence the statement. ■

Asymptotic preserving property. Let us mention an additional property of the proposed splitting integrator: consider the slow-fast system

$$\begin{cases} \mathrm{i}\mathrm{d}u^\epsilon(t) + \Delta u^\epsilon(t)\dfrac{m^\epsilon(t)}{\epsilon}\mathrm{d}t + V[u^\epsilon(t)]u^\epsilon(t)\,\mathrm{d}t = 0, \\ \mathrm{d}m^\epsilon(t) = -\dfrac{1}{\epsilon^2}m^\epsilon(t)\mathrm{d}t + \dfrac{1}{\epsilon}\mathrm{d}\beta(t), \end{cases}$$

where $\epsilon \ll 1$ is a small parameter, with $u^\epsilon(0) = u_0$ and $m^\epsilon(0) = 0$. It is well-known that u^ϵ converges to the solution u of (1.1) when $\epsilon \to 0$ and that the good interpretation of the noise in the limit is indeed the Stratonovich one (see, for instance, [15, Theorem 5.1]). When $\epsilon \ll 1$, one may define the following scheme:

$$\begin{cases} u^\epsilon_{n+1} = e^{\mathrm{i}\frac{\tau m^\epsilon_{n+1}}{\epsilon}\Delta}\left(e^{\mathrm{i}\tau V[u^\epsilon_n]}u^\epsilon_n\right), \\ m^\epsilon_{n+1} = m^\epsilon_n - \dfrac{\tau}{\epsilon^2}m^\epsilon_{n+1} + \dfrac{(\beta(t_{n+1}) - \beta(t_n))}{\epsilon}. \end{cases}$$

This scheme is consistent with the multiscale model above for all values of ϵ when $\tau \to 0$. In addition, it is *asymptotic preserving* (see, for instance, [10, 15] for similar results for stochastic Schrödinger equations, and [4] for the finite-dimensional situation): $u^\epsilon_n \to u_n$ when $\epsilon \to 0$, where u_n is defined by the proposed splitting integrator, which is consistent when $\tau \to 0$ with (1.1). The proposed integrator is thus stable when the white noise $\circ\mathrm{d}\beta(t)$ is approximated by a smoother version $m^\epsilon(t)/\epsilon \mathrm{d}t$ in a diffusion approximation regime.

4.3. Convergence results. We are now in position to state the main result of this article.

Theorem 4.5. *Let $\big(u(t)\big)_{t\geq 0}$, resp., $\big(u_n\big)_{n\in\mathbb{N}_0}$, be the solutions of the stochastic Schrödinger equation (1.1), resp., of the splitting scheme (4.2), with (nonrandom) initial condition u_0.*

Let $m \in \mathbb{N}_0$ and assume that $V \in \mathcal{C}^{m+4}$. For all $p \in [1,\infty)$, all $T \in (0,\infty)$, and all $u_0 \in H^{m+4}$, there exists $C_{m,p}(T, \|u_0\|_{H^{m+4}}) \in (0,\infty)$ such that, for all $\tau \in (0,1)$, one has

$$\sup_{0\leq n\leq N} \left(\mathbb{E}\left[\|u_n - u(t_n)\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq C_{m,p}(T, \|u_0\|_{H^{m+4}})\tau. \tag{4.5}$$

The proof of Theorem 4.5 is postponed to section 5.

Note that contrary to previous works in the literature [15, 1, 6, 7], concerning the analysis of numerical schemes for stochastic Schrödinger equations with white noise dispersion with a globally Lipschitz continuous nonlinearity, in Theorem 4.5 we consider the moments of arbitrary order $p \in [1,\infty)$, instead of only $p = 2$ (mean-square error). We also consider the error in the H^m norm, for arbitrary $m \in \mathbb{N}_0$. We could use the same strategy of proof as in those references when $p = 2$, but we need to use a different strategy when $p \neq 2$ and directly consider the general case $p \in [1,\infty)$.

Remark 4.6. If the initial condition u_0 and the potential V are less regular than in Theorem 4.5, it is possible to obtain the following result: assume that $u_0 \in H^{m+2}$ and that $V \in \mathcal{C}^{m+2}$, then one has

$$\sup_{0\leq n\leq N} \left(\mathbb{E}\left[\|u_n - u(t_n)\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq C_{m,p}(T, \|u_0\|_{H^{m+2}})\tau^{\frac{1}{2}}.$$

Further details are provided in Remark 5.2 below. In particular, we explain which parts of the proof of Theorem 4.5 need to be modified to obtain the above result.

As immediate consequences of the main result of this article we obtain the following corollaries.

Corollary 4.7. *Under the assumptions of Theorem 4.5, one obtains the following error estimate: for all $\varepsilon \in (0,1)$, there exists $C_{m,p,\varepsilon}(T, \|u_0\|_{H^{m+4}}) \in (0,\infty)$ such that, for all $\tau \in (0,1)$, one has*

$$\left(\mathbb{E}\left[\sup_{0\leq n\leq N} \|u_n - u(t_n)\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq C_{m,p,\varepsilon}(T, \|u_0\|_{H^{m+4}})\tau^{1-\varepsilon}. \tag{4.6}$$

Proof. The error estimate (4.6) follows from the first error estimate (4.5) by an elementary argument. Let $\varepsilon \in (0,1)$ and $p \in [1,\infty)$, and choose $q > \max(p, \varepsilon^{-1})$. Using (4.5) one obtains

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq n\leq N} \|u_n - u(t_n)\|_{H^m}^q\right] &\leq \sum_{n=0}^{N} \mathbb{E}\left[\|u_n - u(t_n)\|_{H^m}^q\right] \\
&\leq \frac{T}{\tau}\big(C_{m,q}(T, \|u_0\|_{H^{m+4}})\tau\big)^q \\
&\leq T C_{m,q}(T, \|u_0\|_{H^{m+4}})^q \tau^{q(1-\frac{1}{q})} \\
&\leq C_{m,q}(T, \|u_0\|_{H^{m+4}})\tau^{q(1-\varepsilon)},
\end{aligned}$$

where we recall that $\tau = T/N$. Finally, one obtains (4.6) as follows:

$$\left(\mathbb{E}\left[\sup_{0\le n\le N}\|u_n - u(t_n)\|_{H^m}^p\right]\right)^{\frac{1}{p}} \le \left(\mathbb{E}\left[\sup_{0\le n\le N}\|u_n - u(t_n)\|_{H^m}^q\right]\right)^{\frac{1}{q}}$$
$$\le C_{m,p,\varepsilon}(T, \|u_0\|_{H^{m+4}})\tau^{1-\varepsilon}. \qquad \blacksquare$$

The argument described above gives a slight reduction in the order of convergence from 1 to $1-\varepsilon$, with arbitrarily small $\varepsilon > 0$. It may be possible to obtain (4.6) with $\varepsilon = 0$ using refined arguments in the analysis of the error. To keep the presentation simple, this is not performed in what follows.

The fact that the main error estimate (4.5) holds with arbitrarily large p is important and allows us to choose arbitrarily small ε. If one applies the argument detailed above only when $p = 2$, for instance, one obtains an order of convergence $\frac{1}{2}$ in (4.6).

Corollary 4.8. *Consider the stochastic Schrödinger equation* (1.1) *on the time interval* $[0,T]$ *with solution denoted by* $(u(t))_{t\in[0,T]}$*. Let* u_n *be the numerical solution given by the splitting scheme* (4.2) *with time-step size* τ*. Under the assumptions of Theorem* 4.5*, one has convergence in probability of order one*

$$\lim_{C\to\infty}\sup_{\tau\in(0,1)}\mathbb{P}\left(\|u_N - u(T)\|_{H^m} \ge C\tau\right) = 0,$$

where we recall that $T = N\tau$*.*

Moreover, consider the sequence of time-step sizes given by $\tau_L = \frac{T}{2^L}$*,* $L \in \mathbb{N}$*. Then, for every* $\varepsilon \in (0,1)$*, there exists an almost surely finite random variable* C_ε*, such that for all* $L \in \mathbb{N}$ *one has*

$$\|u_{2^L} - u(T)\|_{H^m} \le C_\varepsilon\left(\frac{T}{2^L}\right)^{1-\varepsilon}.$$

Proof. The result on convergence in probability is a straightforward consequence of Markov's inequality followed by Theorem 4.5:

$$\mathbb{P}\left(\|u_N - u(T)\|_{H^m} \ge C\tau\right) \le \frac{\mathbb{E}\left[\|u_N - u(T)\|_{H^m}\right]}{C\tau} = \frac{C_{m,1}(T, \|u_0\|_{H^{m+4}})}{C} \underset{C\to\infty}{\to} 0.$$

To get the result on almost sure convergence, it suffices to observe that (again by applying Theorem 4.5)

$$\sum_{\ell=0}^{\infty}\frac{\mathbb{E}\left[\|u_{2^\ell} - u(T)\|_{H^m}\right]}{\tau_\ell^{1-\varepsilon}} < \infty,$$

and thus $\frac{\|u_{2^L}-u(T)\|_{H^m}}{\tau_L^{1-\varepsilon}} \underset{L\to\infty}{\to} 0$ almost surely. $\blacksquare$

5. Error analysis: Proof of Theorem 4.5. Before proceeding with the proof of the error estimates (4.5), let us state and prove an auxiliary result on the mappings $\Psi_0(u) = V[u]u$ and $\Psi_\tau(u) = \frac{\Phi_\tau(u)-u}{\mathrm{i}\tau}$.

Lemma 5.1. *Let $m \in \mathbb{N}_0$ and assume that $V \in \mathcal{C}^m$.*
There exists $C_m(V) \in (0,\infty)$ such that for all $u \in H^m$ and all $\tau \in (0,1)$, one has

$$\|\Psi_\tau(u) - \Psi_0(u)\|_{H^m} \leq C_m(V)\tau \left(1 + \|u\|_{L^2}^{\max(4,2m)}\right) \|u\|_{H^m}.$$

Proof. Let us first observe that, by definitions of the operators Ψ_τ and Ψ_0, one has

$$\Psi_\tau(u) - \Psi_0(u) = \Theta_\tau(V[u])u,$$

where $\Theta_\tau(y) = \frac{e^{i\tau y} - 1 - i\tau y}{i\tau}$ for all $y \in \mathbb{R}$.

Applying the inequality from Lemma 2.1, one obtains

$$\|\Psi_\tau(u) - \Psi_0(u)\|_{H^m} \leq C_m \|\Theta_\tau(V[u])\|_{\mathcal{C}^m} \|u\|_{H^m}.$$

It remains to study the behavior of $\|\Theta_\tau(V[u])\|_{\mathcal{C}^m}$.

The auxiliary function Θ_τ satisfies the following properties: for all $k \in \mathbb{N}_0$, there exists $C_k \in (0,\infty)$ such that, for all $y \in \mathbb{R}$, one has

- $|\Theta_\tau(y)| \leq C_0\tau|y|^2$,
- $|\Theta_\tau'(y)| \leq C_1\tau|y|$,
- $|\Theta_\tau^{(k)}(y)| \leq C_k\tau^{k-1} \leq C_k\tau$ for all integers $k \geq 2$ and all $\tau \in (0,1)$.

Using the Faà di Bruno formula, one obtains the bounds

$$\|\partial_\gamma \Theta_\tau(V[u])\|_{\mathcal{C}^0} \leq C \begin{cases} \tau \|V[u]\|_{\mathcal{C}^0}^2 \ , & \gamma = 0, \\ \tau \left(1 + \|V[u]\|_{\mathcal{C}^{|\gamma|}}^{|\gamma|}\right) \ , & \gamma \neq 0. \end{cases}$$

Using the inequality

$$\|V[u]\|_{\mathcal{C}^{|\gamma|}}^{|\gamma|} = \left\|\partial_\gamma V * |u|^2\right\|_{\mathcal{C}^0}^{|\gamma|} \leq C_m(V) \|u\|_{L^2}^{2m},$$

if $|\gamma| \leq m$, then the proof of Lemma 5.1 concludes. ■

We are now in position to give the proof of Theorem 4.5.

Proof of Theorem 4.5. Let us first perform a change of unknowns: for all $t \geq 0$ and $n \in \mathbb{N}_0$, set

$$v(t) = S(0,t)u(t) = S(t,0)^{-1}u(t) \quad \text{and} \quad v_n = S(0,t_n)u_n = S(t_n,0)^{-1}u_n,$$

where $u(t)$ is the solution of (1.1) whereas u_n is defined by the splitting scheme (4.2). Owing to the isometry property (2.2) for the random propagator, one has the equality

$$\|u_n - u(t_n)\|_{H^m} = \|S(t_n,0)\left(v_n - v(t_n)\right)\|_{H^m} = \|v_n - v(t_n)\|_{H^m}$$

for all $m \in \mathbb{N}_0$ and for all $n \in \mathbb{N}_0$. Thus, it is sufficient to prove estimates for the error $E_n = v_n - v(t_n)$.

Using the mild form (3.1) for $u(t)$ and the definition of the splitting scheme (4.2), for all $t \geq 0$ and $n \in \mathbb{N}_0$, one has the following expressions:

$$v(t) = u_0 + \mathrm{i} \int_0^t S(0,s)\Psi_0(u(s))\,\mathrm{d}s,$$
$$v(t_{n+1}) = v(t_n) + \mathrm{i} \int_{t_n}^{t_{n+1}} S(0,t)\Psi_0(u(t))\,\mathrm{d}t,$$
$$v_{n+1} = S(0,t_n)\Phi_\tau(u_n) = v_n + \mathrm{i}\tau S(0,t_n)\Psi_\tau(u_n).$$

The expressions above then give the following decomposition of the error:

$$E_{n+1} = v_{n+1} - v(t_{n+1}) = E_n + \epsilon_n^1 + \epsilon_n^2 + \epsilon_n^3 + \epsilon_n^4 + \epsilon_n^5$$

with local error terms defined by

$$\epsilon_n^1 = \mathrm{i}\tau S(0,t_n)\left(\Psi_\tau(u_n) - \Psi_0(u_n)\right),$$
$$\epsilon_n^2 = \mathrm{i}\tau S(0,t_n)\left(\Psi_0(u_n) - \Psi_0(u(t_n))\right),$$
$$\epsilon_n^3 = \mathrm{i} \int_{t_n}^{t_{n+1}} \left(S(0,t) - S(0,t_n)\right)\left(\Psi_0(u(t_n)) - \Psi_0(u(t))\right)\mathrm{d}t,$$
$$\epsilon_n^4 = \mathrm{i} \int_{t_n}^{t_{n+1}} \left(S(0,t_n) - S(0,t)\right)\Psi_0(u(t_n))\,\mathrm{d}t,$$
$$\epsilon_n^5 = \mathrm{i} \int_{t_n}^{t_{n+1}} S(0,t_n)\left(\Psi_0(u(t_n)) - \Psi_0(u(t))\right)\mathrm{d}t.$$

For $j = 1,\ldots,5$, set $E_n^j = \sum_{k=0}^{n-1} \epsilon_k^j$. Then a straightforward recursion argument yields the equality

$$E_n = \sum_{j=1}^{5} E_n^j = \sum_{j=1}^{5}\sum_{k=0}^{n-1} \epsilon_k^j,$$

and applying Minkowski's inequality one obtains, for $p \in [1,\infty)$,

$$\left(\mathbb{E}\left[\|E_n\|_{H^m}^p\right]\right)^{\frac{1}{p}} \le \sum_{j=1}^{5} \left(\mathbb{E}[\|E_n^j\|_{H^m}^p]\right)^{\frac{1}{p}}.$$

It remains to prove error estimates for $(\mathbb{E}[\left\|E_n^j\right\|_{H^m}^p])^{\frac{1}{p}}$, for $j = 1,\ldots,5$ and $p \in [2,\infty)$ (the case $p \in [1,2)$ is treated using Hölder's inequality). The estimates of the error terms for $j = 1, 2, 3$ follow from straightforward arguments, whereas more work is required to deal with the cases $j = 4$ and $j = 5$ (in order to obtain order of convergence equal to 1, instead of the order 1/2 corresponding to the temporal Hölder regularity of the solution; see equation (3.4)).

We now provide detailed error estimates of those five terms.

• Let us start with the first term. Using Minkowski's inequality and the isometry property (2.2) of the random propagator, one has

$$\begin{aligned}
\left(\mathbb{E}\left[\|E_n^1\|_{H^m}^p\right]\right)^{\frac{1}{p}} &\le \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|\epsilon_k^1\|_{H^m}^p\right]\right)^{\frac{1}{p}} \\
&\le \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|\mathrm{i}\tau S(0,t_k)\left(\Psi_\tau(u_k) - \Psi_0(u_k)\right)\|_{H^m}^p\right]\right)^{\frac{1}{p}} \\
&\le \tau \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|\Psi_\tau(u_k) - \Psi_0(u_k)\|_{H^m}^p\right]\right)^{\frac{1}{p}}.
\end{aligned}$$

Applying Lemma 5.1 and using first the preservation of the L^2-norm property (4.3) for the numerical scheme (Proposition 4.1) and second the almost sure bound (4.4) for the H^m-norm of the numerical solution (Proposition 4.2), one finally obtains, for all $n = 0, \ldots, N$,

$$\begin{aligned}\left(\mathbb{E}\left[\|E_n^1\|_{H^m}^p\right]\right)^{\frac{1}{p}} &\leq C\tau^2 \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\left(1 + \|u_k\|_{L^2}^{\max(4,2m)}\right)^p \|u_k\|_{H^m}^p\right]\right)^{\frac{1}{p}} \\ &\leq C\tau^2 \sum_{k=0}^{n-1} \left(1 + \|u_0\|_{L^2}^{\max(4,2m)}\right)^p \left(\mathbb{E}\left[\|u_k\|_{H^m}^p\right]\right)^{\frac{1}{p}} \\ &\leq C_{m,p}(T, \|u_0\|_{H^m})\tau.\end{aligned}$$

• For the second term, using Minkowski's inequality and the isometry property (2.2) of the random propagator, one has

$$\begin{aligned}\left(\mathbb{E}\left[\|E_n^2\|_{H^m}^p\right]\right)^{\frac{1}{p}} &\leq \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|\epsilon_k^2\|_{H^m}^p\right]\right)^{\frac{1}{p}} \\ &\leq \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|\mathrm{i}\tau S(0,t_k)\left(\Psi_0(u_k) - \Psi_0(u(t_k))\right)\|_{H^m}^p\right]\right)^{\frac{1}{p}} \\ &\leq \tau \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|\Psi_0(u_k) - \Psi_0(u(t_k))\|_{H^m}^p\right]\right)^{\frac{1}{p}}.\end{aligned}$$

Using the local Lipschitz continuity property (2.5) of Ψ_0 (Lemma 2.2), then the almost sure bounds for the H^m norm of the exact solution (equation (3.3) from Proposition 3.1) and of the numerical solution (equation (4.4) from Proposition 4.2), one obtains, for all $n = 0, \ldots, N$,

$$\begin{aligned}\left(\mathbb{E}\left[\|E_n^2\|_{H^m}^p\right]\right)^{\frac{1}{p}} &\leq C\tau \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\left(\|u_k\|_{H^m}^2 + \|u(t_k)\|_{H^m}^2\right)^p \|u_k - u(t_k)\|_{H^m}^p\right]\right)^{\frac{1}{p}} \\ &\leq C(T, \|u_0\|_{H^m})\tau \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|E_k\|_{H^m}^p\right]\right)^{\frac{1}{p}}.\end{aligned}$$

• In order to estimate the third term, applying the Minkowski inequality yields

$$\left(\mathbb{E}\left[\|E_n^3\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|\epsilon_k^3\|_{H^m}^p\right]\right)^{\frac{1}{p}}.$$

Using the inequality (2.3) (combined with the Cauchy–Schwarz inequality) and the local Lipschitz continuity property (2.5) of Ψ_0 (Lemma 2.2) then yields

$$\begin{aligned}&\left(\mathbb{E}\left[\|\epsilon_k^3\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq \int_{t_k}^{t_{k+1}} \left(\mathbb{E}\left[\|\left(S(0,t_k) - S(0,t)\right)\left(\Psi_0(u(t)) - \Psi_0(u(t_k))\right)\|_{H^m}^p\right]\right)^{\frac{1}{p}} \mathrm{d}t \\ &\leq C\tau^{\frac{1}{2}} \int_{t_k}^{t_{k+1}} \left(\mathbb{E}\left[\|\Psi_0(u(t)) - \Psi_0(u(t_k))\|_{H^{m+2}}^{2p}\right]\right)^{\frac{1}{2p}} \mathrm{d}t \\ &\leq C\tau^{\frac{1}{2}} \int_{t_k}^{t_{k+1}} \left(\mathbb{E}\left[\left(\|u(t)\|_{H^{m+2}}^2 + \|u(t_k)\|_{H^{m+2}}^2\right)^{2p} \|u(t) - u(t_k)\|_{H^{m+2}}^{2p}\right]\right)^{\frac{1}{2p}} \mathrm{d}t.\end{aligned}$$

Using the almost sure bound (3.3) for the H^{m+2}-norm of the exact solution (Proposition 3.1) and the temporal regularity estimate (3.4), one finally obtains

$$\left(\mathbb{E}\left[\|\epsilon_k^3\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq C(T,\|u_0\|_{H^{m+4}})\tau^{\frac{1}{2}}\int_{t_k}^{t_{k+1}}(t-t_k)^{\frac{1}{2}}\,\mathrm{d}t \leq C(T,\|u_0\|_{H^{m+4}})\tau^2,$$

and finally one has, for all $n=0,\dots,N$,

$$\left(\mathbb{E}\left[\|E_n^3\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq \sum_{k=0}^{n-1}\left(\mathbb{E}\left[\|\epsilon_k^3\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq C_{m,p}(T,\|u_0\|_{H^{m+4}})\tau.$$

• Let us now focus on the the fourth term. As explained above, one needs to be careful to obtain an order of convergence equal to 1. Indeed, for the fourth term, applying (2.3) directly (and appropriate bounds) would only give order of convergence 1/2 of the splitting scheme.

Let us define auxiliary processes: for all $n\in\mathbb{N}_0$ and all $t\in[t_n,t_{n+1}]$, set

$$v_n(t)=S(0,t)\Psi_0(u(t_n)).$$

Note that $v_n(t)=S(t_n,t)v_n(t_n)=\mathrm{e}^{-\mathrm{i}(\beta(t)-\beta(t_n))\Delta}v_n(t_n)$ for all $t\in[t_n,t_{n+1}]$. As a consequence, the process $(v_n(t))_{t\in[t_n,t_{n+1}]}$ is the solution of the linear stochastic evolution equation

$$\mathrm{d}v_n(t)=-\mathrm{i}\Delta v_n(t)\circ\mathrm{d}\beta(t)=-\mathrm{i}\Delta v_n(t)\,\mathrm{d}\beta(t)-\frac{1}{2}\Delta^2 v_n(t)\,\mathrm{d}t$$

with $v_n(t_n)=S(0,t_n)\Psi_0(u(t_n))$.

The local error term ϵ_n^4 is rewritten as follows in terms of the auxiliary process v_n:

$$\begin{aligned}
\epsilon_n^4 &= \mathrm{i}\int_{t_n}^{t_{n+1}}\left(S(0,t_n)-S(0,t)\right)\Psi_0(u(t_n))\,\mathrm{d}t\\
&= \mathrm{i}\int_{t_n}^{t_{n+1}}\left(v_n(t_n)-v_n(t)\right)\,\mathrm{d}t\\
&= \int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\Delta v_n(s)\,\mathrm{d}\beta(s)\,\mathrm{d}t-\frac{\mathrm{i}}{2}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\Delta^2 v_n(s)\,\mathrm{d}s\,\mathrm{d}t\\
&= \epsilon_n^{4,1}+\epsilon_n^{4,2}
\end{aligned}$$

with

$$\begin{aligned}
\epsilon_n^{4,1} &= \int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\Delta v_n(s)\,\mathrm{d}\beta(s)\,\mathrm{d}t\\
\epsilon_n^{4,2} &= -\frac{\mathrm{i}}{2}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\Delta^2 v_n(s)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Set also $E_n^{4,1}=\sum_{k=0}^{n-1}\epsilon_k^{4,1}$ and $E_n^{4,2}=\sum_{k=0}^{n-1}\epsilon_k^{4,2}$. Using Minkowski's inequality, one then gets

$$\left(\mathbb{E}\left[\|E_n^4\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq \left(\mathbb{E}\left[\|E_n^{4,1}\|_{H^m}^p\right]\right)^{\frac{1}{p}}+\left(\mathbb{E}\left[\|E_n^{4,2}\|_{H^m}^p\right]\right)^{\frac{1}{p}}.$$

On the one hand, observe that applying the stochastic Fubini theorem gives the equality

$$\epsilon_n^{4,1} = \int_{t_n}^{t_{n+1}} \int_s^{t_{n+1}} \Delta v_n(s)\,\mathrm{d}t\,\mathrm{d}\beta(s) = \int_{t_n}^{t_{n+1}} (t_{n+1}-s)\Delta v_n(s)\,\mathrm{d}\beta(s).$$

Introduce the (adapted) auxiliary process $\overline{v}$, such that $\overline{v}(s) = (t_{k+1}-s)v_k(s)$ for all $s \in [t_k, t_{k+1}]$ and $k = 0, \dots, N-1$. Then the error term $E_n^{4,1}$ is rewritten as the Itô integral

$$E_n^{4,1} = \int_0^{t_n} \Delta\overline{v}(s)\,\mathrm{d}\beta(s),$$

and applying the Burkholder–Davis–Gundy and Hölder inequalities, for all $p \geq 2$, one obtains

$$\begin{aligned}\left(\mathbb{E}\left[\|E_n^{4,1}\|_{H^m}^p\right]\right)^{\frac{1}{p}} &= \left(\mathbb{E}\left[\left\|\int_0^{t_n} \Delta\overline{v}(s)\,\mathrm{d}\beta(s)\right\|_{H^m}^p\right]\right)^{\frac{1}{p}}\\ &\leq C_p(T)\left(\int_0^{t_n} \mathbb{E}\left[\|\Delta\overline{v}(s)\|_{H^m}^p\right]\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\leq C_p(T)\left(\int_0^{t_n} \mathbb{E}\left[\|\overline{v}(s)\|_{H^{m+2}}^p\right]\,\mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned}$$

By the definitions of $\overline{v}(s)$ and of $v_k(s)$, one obtains

$$\begin{aligned}\int_0^{t_n} \mathbb{E}\left[\|\overline{v}(s)\|_{H^{m+2}}^p\right]\,\mathrm{d}s &= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \mathbb{E}\left[\|(t_{k+1}-s)v_k(s)\|_{H^{m+2}}^p\right]\,\mathrm{d}s\\ &\leq \tau^p \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \mathbb{E}\left[\|S(0,s)\Psi_0(u(t_k))\|_{H^{m+2}}^p\right]\,\mathrm{d}s\\ &\leq \tau^{p+1}\sum_{k=0}^{n-1}\mathbb{E}\left[\|\Psi_0(u(t_k))\|_{H^{m+2}}^p\right]\\ &\leq C(T, \|u_0\|_{H^{m+2}})\tau^p,\end{aligned}$$

using the isometry property (2.2), the inequality (2.4), and the exact preservation of the L^2-norm (3.2) as well as the almost sure bound (3.3) for the H^{m+2} norm of the exact solution (see Proposition 3.1).

On the other hand, for the second term, using Minkowski's inequality and the definition of the auxiliary processes v_k, one obtains

$$\begin{aligned}\left(\mathbb{E}\left[\|E_n^{4,2}\|_{H^m}^p\right]\right)^{\frac{1}{p}} &\leq \sum_{k=0}^{n-1}\left(\mathbb{E}\left[\left\|\epsilon_k^{4,2}\right\|_{H^m}^p\right]\right)^{\frac{1}{p}}\\ &\leq \frac{1}{2}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}\left(\mathbb{E}\left[\|v_k(s)\|_{H^{m+4}}^p\right]\right)^{\frac{1}{p}}\,\mathrm{d}s\,\mathrm{d}t\\ &\leq C\tau^2\sum_{k=0}^{n-1}\left(\mathbb{E}\left[\|S(0,t_k)\Psi_0(u(t_k))\|_{H^{m+4}}^p\right]\right)^{\frac{1}{p}}\\ &\leq C(T, \|u_0\|_{H^{m+4}})\tau,\end{aligned}$$

using the isometry property (2.2), the inequality (2.4), and the almost sure bound (3.3) for the H^{m+4} norm of the exact solution.

Gathering the estimates, one obtains the following estimate for the fourth error term: for all $n = 0, \ldots, N$,

$$\left(\mathbb{E}\left[\|E_n^4\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq C_{m,p}(T, \|u_0\|_{H^{m+4}})\tau.$$

• It remains to deal with the fifth error term. Using a second order Taylor expansion, one has, for $t \in [t_n, t_{n+1}]$,

$$\begin{aligned}\Psi_0(u(t)) - \Psi_0(u(t_n)) &= \Psi_0'(u(t_n)).\,(u(t) - u(t_n)) \\ &\quad + \int_0^1 (1-\xi)\Psi_0''((1-\xi)u(t_n) + \xi u(t)).(u(t) - u(t_n), u(t) - u(t_n))\,\mathrm{d}\xi \\ &= \Psi_0'(u(t_n)).\left(\left(S(t,t_n) - I\right)u(t_n) + \mathrm{i}\int_{t_n}^t S(t,s)\Psi_0(u(s))\,\mathrm{d}s\right) + R_n(t),\end{aligned}$$

using the mild formulation (3.1) for the exact solution, where one has defined the quantity $R_n(t) = \int_0^1 (1-\xi)\Psi_0''((1-\xi)u(t_n) + \xi u(t)).(u(t) - u(t_n), u(t) - u(t_n))\,\mathrm{d}\xi$.

For all $n = 0, \ldots, N-1$, set

$$\begin{aligned}\epsilon_n^{5,1} &= -\mathrm{i}\int_{t_n}^{t_{n+1}} S(0,t_n)\Psi_0'(u(t_n)).\left(\left(S(t,t_n) - I\right)u(t_n)\right)\mathrm{d}t \\ \epsilon_n^{5,2} &= \int_{t_n}^{t_{n+1}} S(0,t_n)\Psi_0'(u(t_n)).\left(\int_{t_n}^t S(t,s)\Psi_0(u(s))\,\mathrm{d}s\right)\mathrm{d}t \\ \epsilon_n^{5,3} &= -\mathrm{i}\int_{t_n}^{t_{n+1}} S(0,t_n)R_n(t)\,\mathrm{d}t\end{aligned}$$

and $E_n^{5,j} = \sum_{k=0}^{n-1} \epsilon_n^{5,j}$, $j = 1,2,3$. Note that $\epsilon_n^5 = \epsilon_n^{5,1} + \epsilon_n^{5,2} + \epsilon_n^{5,3}$ and $E_n^5 = E_n^{5,1} + E_n^{5,2} + E_n^{5,3}$ for all $n = 0, \ldots, N-1$, and Minkowski's inequality yields

$$\left(\mathbb{E}\left[\|E_n^5\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq \left(\mathbb{E}\left[\|E_n^{5,1}\|_{H^m}^p\right]\right)^{\frac{1}{p}} + \left(\mathbb{E}\left[\|E_n^{5,2}\|_{H^m}^p\right]\right)^{\frac{1}{p}} + \left(\mathbb{E}\left[\|E_n^{5,3}\|_{H^m}^p\right]\right)^{\frac{1}{p}}.$$

It remains to obtain estimates for each of the three error terms in the right-hand side above.

(i) To treat the first error terms $E_n^{5,1}$ and $\epsilon_n^{5,1}$, one follows the same strategy as for the error terms $E_n^{4,1}$ and $\epsilon_n^{4,1}$ above. Let us define auxiliary processes: for all $n \in \mathbb{N}_0$ and $t \in [t_n, t_{n+1}]$, set

$$w_n(t) = S(t,t_n)u(t_n).$$

For each $n \in \mathbb{N}_0$, the process $(w_n(t))_{t\in[t_n,t_{n+1}]}$ is the solution of the linear stochastic evolution

$$\mathrm{d}w_n(t) = \mathrm{i}\Delta w_n(t) \circ \mathrm{d}\beta(t) = \mathrm{i}\Delta w_n(t)\,\mathrm{d}\beta(s) - \frac{1}{2}\Delta^2 w_n(t)\,\mathrm{d}t$$

with initial value $w_n(t_n) = u(t_n)$ (see the definition (2.1) of the random propagator $S(t,s)$). The local error term $\epsilon_n^{5,1}$ is rewritten as follows in terms of the auxiliary process w_n:

$$\begin{aligned}\epsilon_n^{5,1} &= -\mathrm{i}\int_{t_n}^{t_{n+1}} S(0,t_n)\Psi_0'(u(t_n)).\left(S(t,t_n)-I\right)u(t_n)\,\mathrm{d}t\\
&= -\mathrm{i}\int_{t_n}^{t_{n+1}} S(0,t_n)\Psi_0'(u(t_n)).\left(w_n(t)-w_n(t_n)\right)\,\mathrm{d}t\\
&= \int_{t_n}^{t_{n+1}} S(0,t_n)\Psi_0'(u(t_n)).\int_{t_n}^{t}\Delta w_n(s)\,\mathrm{d}\beta(s)\,\mathrm{d}t\\
&+\frac{\mathrm{i}}{2}\int_{t_n}^{t_{n+1}} S(0,t_n)\Psi_0'(u(t_n)).\int_{t_n}^{t}\Delta^2 w_n(s)\,\mathrm{d}s\,\mathrm{d}t\\
&= \epsilon_n^{5,1,1}+\epsilon_n^{5,1,2}\end{aligned}$$

with

$$\begin{aligned}\epsilon_n^{5,1,1} &= \int_{t_n}^{t_{n+1}} S(0,t_n)\Psi_0'(u(t_n)).\int_{t_n}^{t}\Delta w_n(s)\,\mathrm{d}\beta(s)\,\mathrm{d}t\\
\epsilon_n^{5,1,2} &= \frac{\mathrm{i}}{2}\int_{t_n}^{t_{n+1}} S(0,t_n)\Psi_0'(u(t_n)).\int_{t_n}^{t}\Delta^2 w_n(s)\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Set also $E_n^{5,1,1}=\sum_{k=0}^{n-1}\epsilon_k^{5,1,1}$ and $E_n^{5,1,2}=\sum_{k=0}^{n-1}\epsilon_k^{5,1,2}$. Using Minkowski's inequality, one then gets

$$\left(\mathbb{E}\left[\|E_n^{5,1}\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq \left(\mathbb{E}\left[\|E_n^{5,1,1}\|_{H^m}^p\right]\right)^{\frac{1}{p}} + \left(\mathbb{E}\left[\|E_n^{5,1,2}\|_{H^m}^p\right]\right)^{\frac{1}{p}}.$$

On the one hand, observe that applying the stochastic Fubini theorem gives the equality

$$\begin{aligned}\epsilon_n^{5,1,1} &= \int_{t_n}^{t_{n+1}} S(0,t_n)\Psi_0'(u(t_n)).\int_{t_n}^{t}\Delta w_n(s)\,\mathrm{d}\beta(s)\,\mathrm{d}t\\
&= S(0,t_n)\Psi_0'(u(t_n)).\int_{t_n}^{t_{n+1}}\int_{s}^{t_{n+1}}\Delta w_n(s)\,\mathrm{d}t\,\mathrm{d}\beta(s)\\
&= S(0,t_n)\Psi_0'(u(t_n)).\int_{t_n}^{t_{n+1}}(t_{n+1}-s)\Delta w_n(s)\,\mathrm{d}\beta(s).\end{aligned}$$

Introduce the (adapted) auxiliary process $\overline{w}$, such that, for all $s\in[t_k,t_{k+1}]$ and $k=0,\dots,N-1$, one has $\overline{w}(s)=(t_{k+1}-s)S(0,t_k)\Psi_0'(u(t_k)).(\Delta w_k(s))$. Then the error term $E_n^{5,1,1}$ is rewritten as the Itô integral

$$E_n^{5,1,1}=\int_0^{t_n}\overline{w}(s)\,\mathrm{d}\beta(s),$$

and applying the Burkholder–Davis–Gundy and Hölder inequalities, for all $p\geq 2$, one obtains

$$\begin{aligned}\left(\mathbb{E}\left[\|E_n^{5,1,1}\|_{H^m}^p\right]\right)^{\frac{1}{p}} &= \left(\mathbb{E}\left[\left\|\int_0^{t_n}\overline{w}(s)\,\mathrm{d}\beta(s)\right\|_{H^m}^p\right]\right)^{\frac{1}{p}}\\
&\leq C_p(T)\left(\int_0^{t_n}\mathbb{E}\left[\|\overline{w}(s)\|_{H^m}^p\right]\,\mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned}$$

By the definitions of $\overline{w}(s)$ and of $w_k(s)$, one obtains

$$\begin{aligned}\int_0^{t_n} \mathbb{E}\left[\|\overline{w}(s)\|_{H^m}^p\right] \mathrm{d}s &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[\left\|(t_{k+1}-s)S(0,t_k)\Psi_0'(u(t_k)).\Delta w_k(s)\right\|_{H^m}^p\right] \mathrm{d}s \\ &\le C\tau^p \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[\left\|\Psi_0'(u(t_k)).\Delta S(s,t_k)u(t_k)\right\|_{H^m}^p\right] \mathrm{d}s \\ &\le C\tau^{p+1} \sum_{k=0}^{n-1} \mathbb{E}\left[\|u(t_k)\|_{H^m}^{2p} \|\Delta u(t_k)\|_{H^m}^p\right] \\ &\le C\tau^{p+1} \sum_{k=0}^{n-1} \mathbb{E}\left[\|u(t_k)\|_{H^{m+2}}^{3p}\right] \\ &\le C(T, \|u_0\|_{H^{m+2}})\tau^p,\end{aligned}$$

using the isometry property (2.2), the inequality (2.6), and the almost sure bound (3.3) for the H^{m+2} norm of the exact solution.

On the other hand, for the second term, using Minkowski's inequality and the definition of the auxiliary processes w_k, one obtains

$$\begin{aligned}&\left(\mathbb{E}\left[\|E_n^{5,1,2}\|_{H^m}^p\right]\right)^{\frac{1}{p}} \le \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\left\|\epsilon_k^{5,1,2}\right\|_{H^m}^p\right]\right)^{\frac{1}{p}} \\ &\le \frac{1}{2} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \left(\mathbb{E}\left[\left\|S(0,t_k)\Psi_0'(u(t_k)).\Delta^2 S(s,t_k)u(t_k)\right\|_{H^m}^p\right]\right)^{\frac{1}{p}} \mathrm{d}s\,\mathrm{d}t \\ &\le C\tau^2 \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|u(t_k)\|_{H^m}^{2p} \left\|\Delta^2 u(t_k)\right\|_{H^m}^p\right]\right)^{\frac{1}{p}} \\ &\le C\tau^2 \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|u(t_k)\|_{H^{m+4}}^{3p}\right]\right)^{\frac{1}{p}} \\ &\le C(T, \|u_0\|_{H^{m+4}})\tau,\end{aligned}$$

using the isometry property (2.2), the inequality (2.4), and the almost sure bound (3.3) for the H^{m+4} norm of the exact solution.

Gathering the estimates for $E_n^{5,1,1}$ and $E_n^{5,1,2}$, one finally obtains, for all $n = 0, \dots, N$,

$$\left(\mathbb{E}\left[\left\|E_n^{5,1}\right\|_{H^m}^p\right]\right)^{\frac{1}{p}} \le C_{m,p}(T, \|u_0\|_{H^{m+4}})\tau.$$

(ii) To deal with the error terms $E_n^{5,2}$ and $\epsilon_n^{5,2}$, using Minkowski's inequality, the isometry property (2.2), and the inequalities (2.6) and (2.4) (Lemma 2.2), one obtains

$$\begin{aligned}&\left(\mathbb{E}\left[\|E_n^{5,2}\|_{H^m}^p\right]\right)^{\frac{1}{p}} \le \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\left\|\epsilon_k^{5,2}\right\|_{H^m}^p\right]\right)^{\frac{1}{p}} \\ &\le \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \left(\mathbb{E}\left[\left\|S(0,t_k)\Psi_0'(u(t_k)).\left(S(t,s)\Psi_0(u(s))\right)\right\|_{H^m}^p\right]\right)^{\frac{1}{p}} \mathrm{d}s\,\mathrm{d}t\end{aligned}$$

$$\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \left(\mathbb{E}\left[\|u(t_k)\|_{H^m}^{2p} \|\Psi_0(u(s))\|_{H^m}^{p}\right]\right)^{\frac{1}{p}} \,\mathrm{d}s\,\mathrm{d}t$$

$$\leq C\tau \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(\mathbb{E}\left[\|u(t_k)\|_{H^m}^{2p} \|u(s)\|_{L^2}^{2p} \|u(s)\|_{H^m}^{p}\right]\right)^{\frac{1}{p}} \,\mathrm{d}s.$$

Finally, using the almost sure bound (3.3) for the H^m norm of the exact solution, one obtains, for all $n = 0, \ldots, N$,

$$\left(\mathbb{E}\left[\left\|E_n^{5,2}\right\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq C_{m,p}(T, \|u_0\|_{H^m})\tau.$$

(iii) To deal with the error terms $E_n^{5,3}$ and $\epsilon_n^{5,3}$, using Minkowski's inequality gives

$$\left(\mathbb{E}\left[\|E_n^{5,3}\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\left\|\epsilon_k^{5,3}\right\|_{H^m}^p\right]\right)^{\frac{1}{p}}$$

$$\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(\mathbb{E}\left[\|S(0,t_k)R_k(t)\|_{H^m}^p\right]\right)^{\frac{1}{p}} \,\mathrm{d}t.$$

Using the isometry property (2.2) and the inequality (2.7) (Lemma 2.2), one obtains

$$\left(\mathbb{E}\left[\left\|E_n^{5,3}\right\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq C \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(\mathbb{E}\left[(\|u(t)\|_{H^m} + \|u(t_k)\|_{H^m})^p \|u(t) - u(t_k)\|_{H^m}^{2p}\right]\right)^{\frac{1}{p}} \,\mathrm{d}t$$

$$\leq C_p(T, \|u_0\|_{H^m}) \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(\mathbb{E}\left[\|u(t) - u(t_k)\|_{H^m}^{2p}\right]\right)^{\frac{1}{p}} \,\mathrm{d}t$$

$$\leq C_{m,p}(T, \|u_0\|_{H^{m+2}})\tau,$$

where the almost sure bound (3.3) for the H^m norm of the exact solution and the temporal regularity estimate (3.4) (Proposition 3.1) have been used.

Gathering the estimates, one finally obtains the last required result: for all $n = 0, \ldots, N$

$$\left(\mathbb{E}\left[\left\|E_n^5\right\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq C_{m,p}(T, \|u_0\|_{H^{m+4}})\tau.$$

• We are now in position to obtain the error estimate (4.5). Gathering the previously obtained estimates, one has, for all $n = 0, \ldots, N$,

$$\left(\mathbb{E}\left[\|E_n\|_{H^m}^p\right]\right)^{\frac{1}{p}} \leq \sum_{j=1}^{5} \left(\mathbb{E}\left[\left\|E_n^j\right\|_{H^m}^p\right]\right)^{\frac{1}{p}}$$

$$\leq C_{m,p}(T, \|u_0\|_{H^{m+4}})\tau + C(T, \|u_0\|_{H^m})\tau \sum_{k=0}^{n-1} \left(\mathbb{E}\left[\|E_k\|_{H^m}^p\right]\right)^{\frac{1}{p}}.$$

Applying the discrete Gronwall lemma then gives (4.5) and concludes the proof of Theorem 4.5. ∎

Remark 5.2. Let us explain the main modifications that are required in the proof above to obtain the result stated in Remark 4.6: if $u_0 \in H^{m+2}$ and $V \in \mathcal{C}^{m+2}$, the strong order of convergence is $1/2$ instead of 1.

The treatments of the error terms E_n^1 and E_n^2 are not modified, but the other terms need to be estimated differently since it is not possible anymore to use estimates for the H^{m+4} norm with the weakened regularity conditions on u_0 and V.

To estimate the error term E_n^3, since $u(t)$ and $u(t_k)$ only belong to H^{m+2}, one can only use the upper bound $\|u(t) - u(t_k)\|_{H^{m+2}} \le \|u(t)\|_{H^{m+2}} + \|u(t_k)\|_{H^{m+2}}$ and the almost sure bound (3.3) of the H^{m+2} norm, instead of using the temporal regularity estimate (3.4).

The treatment of the error term E_n^4 is subtantially simplified: to estimate ϵ_n^4, it suffices to exploit the inequality (2.3), the almost sure bound (3.3) of the H^{m+2} norm, and the inequality $\mathbb{E}\left[|\beta(t) - \beta(t_n)|^{2p}\right] \le C_p \tau^p$ for $|t - t_n| \le \tau$.

Finally, the treatment of the error term E_n^5 is also substantially simplified: it suffices to employ the local Lipschitz continuity property (2.5) of Ψ_0 and the temporal regularity property (3.4). One concludes using the almost sure bounds (3.3) for the H^{m+2} norm.

6. Numerical experiments.

We present some numerical experiments in dimensions 1 and 2 in order to support and illustrate the above theoretical results. In addition, we shall compare the behavior of the splitting scheme (4.2) (denoted by SPLIT below) with the following time integrators for the stochastic nonlinear Schrödinger equation (1.1):

- the stochastic exponential integrator from [6] (adapted to the present nonlocal interaction cubic nonlinearity, denoted by EXP)

$$u_{n+1} = S(t_{n+1}, t_n)u_n + \mathrm{i}\tau S(t_{n+1}, t_n)V[u_n]u_n,$$

- the semi-implicit midpoint scheme (denoted MID)

$$\mathrm{i}\frac{u_{n+1} - u_n}{\tau} + \frac{\chi_n}{\sqrt{\tau}}\Delta u_{n+1/2} + V[u_n]u_n = 0,$$

  where $u_{n+1/2} = \frac{1}{2}(u_n + u_{n+1})$ and $\chi_n = \frac{\beta(t_{n+1}) - \beta(t_n)}{\sqrt{\tau}}$. This is a modification of the Crank–Nicolson from [1] for the nonlinear interaction nonlinearity studied here.

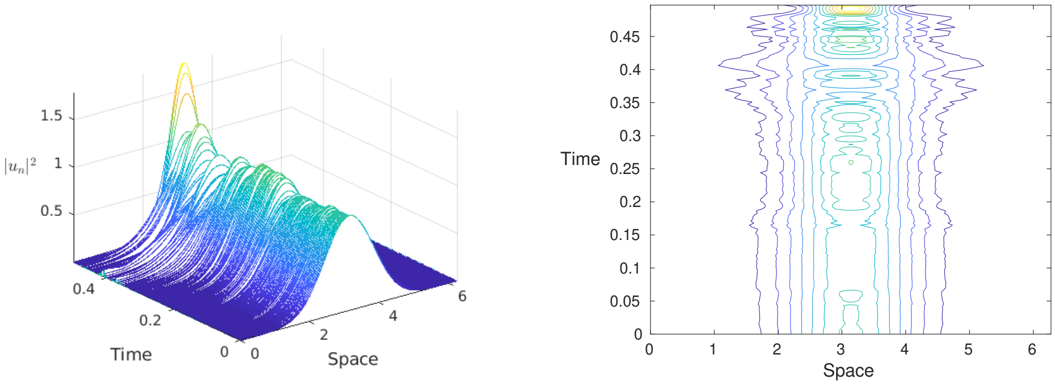
For the numerical experiments in dimension 1, unless stated otherwise, we consider the SPDE (1.1) with the potential $V(x) = \cos(x)$ on the one-dimensional torus with periodic boundary conditions $[0, 2\pi]$. The spatial discretization is done by a pseudospectral method with M Fourier modes. The initial value is given by $u_0(x) = \exp(-0.5(x - \pi)^2)$. Parameters for the numerical experiments in dimension 2 are provided in the dedicated subsection.

6.1. Evolution plots in dimension 1 (1d).

To illustrate the interplay and the balance between the random dispersion and the nonlinearity, in Figure 1, we display the evolution of $|u_n|^2$ along one sample of the numerical solution obtained by the splitting integrator (4.2). The discretization parameters are $\tau = 2^{-14}$ and $M = 2^{10}$, and the time interval is given by $[0, 0.5]$.

6.2. Conservation of the L^2-norm in dimension 1.

It is known that the L^2-norm (or mass) of the solution to the SPDE (1.1) remains constant for all times (see Proposition 3.1).



(a) Space-time evolution (b) Contour plot

Figure 1. *Space-time evolution in* 1d *and contour plot for the splitting integrator* (4.2). *The discretization parameters are* $\tau = 2^{-14}$ *and* $M = 2^{10}$ *Fourier modes.*

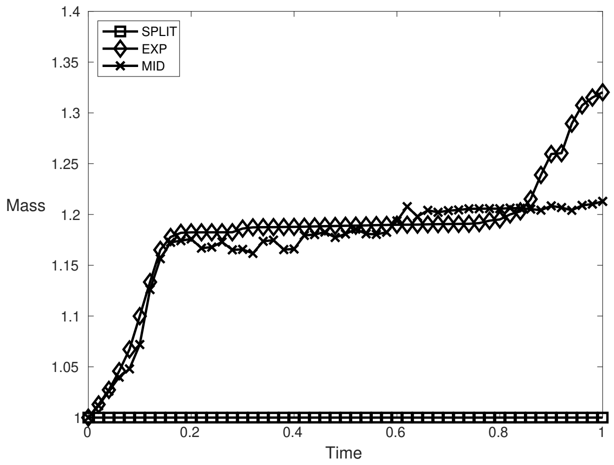


Figure 2. *Preservation of the* L^2*-norm in* 1d*: Splitting scheme* (□)*, exponential integrator* (♢)*, and midpoint scheme* (×)*.*

Figure 2 illustrates the corresponding behavior of the above numerical integrators along one sample path. For this numerical experiment, we consider the parameters $\tau = 2^{-8}$ and $M = 2^{10}$ Fourier modes and the time interval $[0, 1]$. The initial value is as in the previous numerical experiment, but normalized so that its L^2-norm is one. Exact preservation of the L^2-norm for the splitting scheme is observed, as stated in Proposition 4.1, whereas a small drift is observed for the exponential integrator and the midpoint scheme.

6.3. Strong convergence in dimension 1. We now illustrate the mean-square convergence of the splitting scheme (4.2) stated in Theorem 4.5. $M = 2^{10}$ Fourier modes are used for the spatial discretization. The mean-square errors

$$\mathbb{E}[\|u_{\text{ref}}(x, T_{\text{end}}) - u_N(x)\|_{H^1}^2]^{1/2}$$

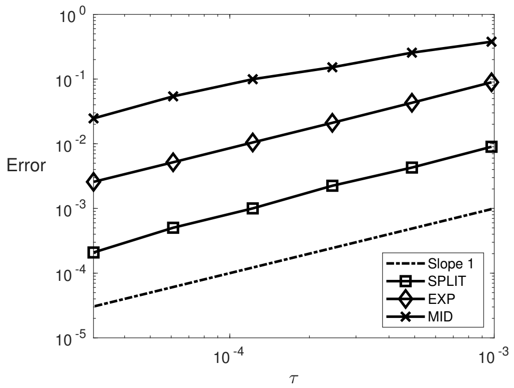


Figure 3. *Mean-square errors in 1d as a function of the time step: Splitting scheme (□), exponential integrator (◊), and midpoint scheme (×). The dotted line has slope 1.*

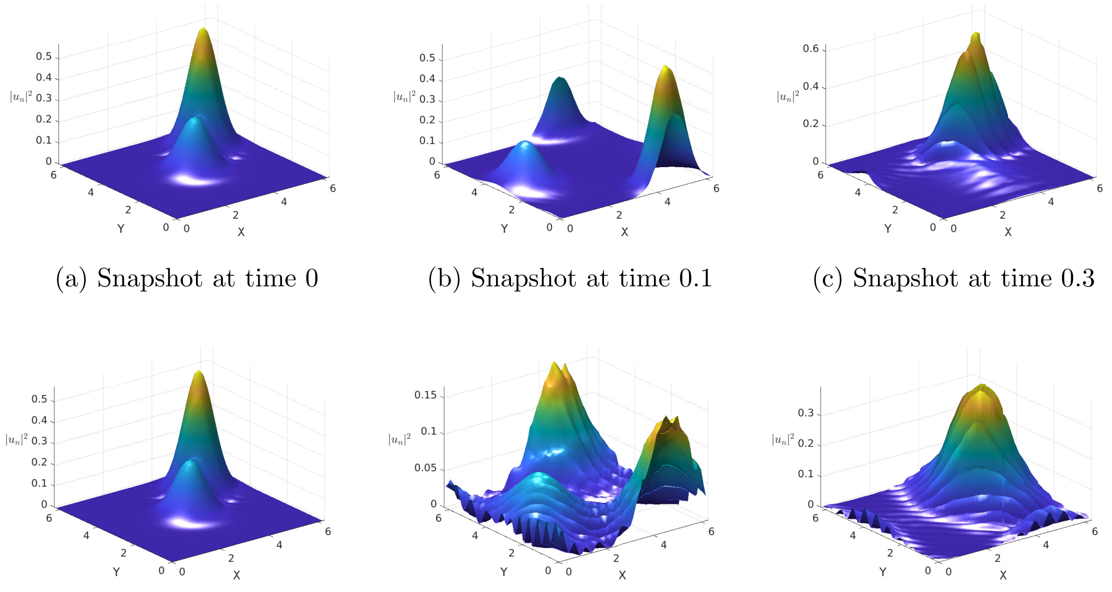
at time $T_{\text{end}} = 1$ are displayed in Figure 3 for various values of the time step $\tau = 2^{-\ell}$ for $\ell = 10, \dots, 16$. Here, we simulate the reference solution $u_{\text{ref}}(x,t)$ with the splitting scheme, with a small time step $\tau_{\text{ref}} = 2^{-18}$. The expected values are approximated by computing averages over $M_s = 100$ samples. In Figure 3, we observe convergence of order 1 for all time integrators. Note that the strong order of convergence of the exponential scheme and midpoint integrator are not known in the case of the considered nonlocal interaction potential, whereas Figure 3 illustrates our main result, Theorem 4.5, for the splitting scheme.

6.4. Numerical experiments in dimension 2 (2d). Let us first compare the evolution of the numerical solution given by the proposed splitting scheme (4.2) when applied to the SPDE (1.1) as well as a deterministic version of it. In Figure 4, we present snapshots of the numerical solutions obtained by the splitting integrator. The SPDE is considered in the spatial domain $[0,2\pi]\times[0,2\pi]$ and on the time interval $[0,0.3]$. We take the following initial condition $u_0(x,y) = 0.5\exp(-0.9((x-\pi)^2+(y-\pi)^2))\exp(-i10x)+0.75\exp(-((x-3\pi/2)^2+(y-3\pi/2)^2))\exp(i10y)$ and the discretization parameters are $\tau = 10^{-4}$ and $M = 2^8$.

Next, we illustrate the preservation of the L^2-norm (or mass) of solutions to the SPDE (1.1) (see Proposition 3.1). Figure 5 illustrates the evolution of this quantity along one sample path of the splitting integrator (4.2), the stochastic exponential integrator, as well as the midpoint scheme. For this numerical experiment, we consider the same parameters as above and take $\tau = 10^{-4}$ and $M = 2^8$ Fourier modes and the time interval $[0,1]$. Note that the initial value is the same as in the previous numerical experiment, but normalized so that its L^2-norm is equal to one. As stated in Proposition 4.1, exact preservation of the L^2-norm for the splitting scheme is observed. As in the corresponding numerical experiment in 1d, the L^2-norm is seen not to be preserved for the exponential integrator and the midpoint scheme.

We conclude the paper by illustrating the mean-square convergence of the splitting scheme (4.2) stated in Theorem 4.5. We take $M = 2^7$ Fourier modes for the spatial discretization and compute the mean-square errors

$$\mathbb{E}[\|u_{\text{ref}}(\cdot,T_{\text{end}}) - u_N(\cdot)\|_{H^1}^2]^{1/2}$$



(a) Snapshot at time 0 (b) Snapshot at time 0.1 (c) Snapshot at time 0.3

(d) Snapshot at time 0 (e) Snapshot at time 0.1 (f) Snapshot at time 0.3

Figure 4. *Snapshots of the evolution of the splitting integrator in 2d (up: deterministic case; down: stochastic case).*

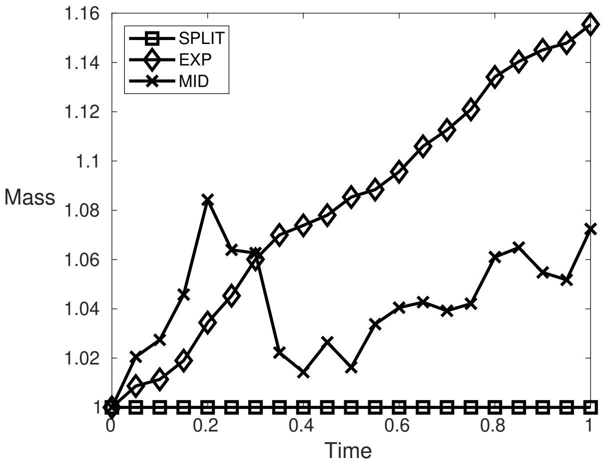


Figure 5. *Evolution of the L^2-norm in 2d: Splitting scheme ($\square$), exponential integrator ($\Diamond$), and midpoint scheme ($\times$).*

at time $T_{\text{end}} = 1$ for the time steps $\tau = 2^{-\ell}$ for $\ell = 10, \dots, 17$. Here, we simulate the reference solution u_{ref} with the splitting scheme, with a small time step $\tau_{\text{ref}} = 2^{-17}$. The expected values are approximated by computing averages over $M_s = 250$ samples. The results are presented in Figure 6, where one can observe the order 1 of convergence of the splitting scheme.

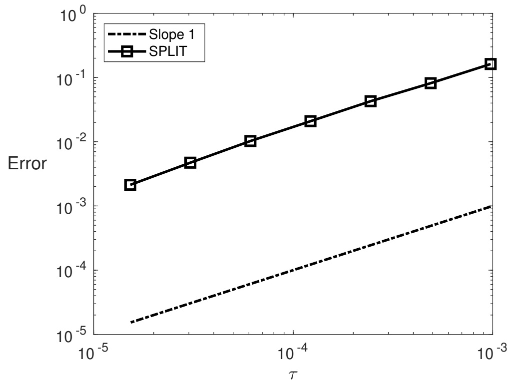


Figure 6. *Mean-square errors in* 2d *as a function of the time step: splitting scheme (□). The dotted line has slope* 1.

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