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Weak convergence rates of splitting schemes for the stochastic Allen–Cahn equation

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Abstract
This article is devoted to the analysis of the weak rates of convergence of schemes introduced by the authors in a recent work, for the temporal discretization of the one-dimensional stochastic Allen–Cahn equation driven by space-time white noise. The schemes are based on splitting strategies and are explicit. We prove that they have a weak rate of convergence equal to $\frac{1}{2}$, like in the more standard case of SPDEs with globally Lipschitz continuous nonlinearity. To deal with the polynomial growth of the nonlinearity, several new estimates and techniques are used. In particular, new regularity results for solutions of related infinite dimensional Kolmogorov equations are established. Our contribution is the first one in the literature concerning weak convergence rates for parabolic semilinear SPDEs with non globally Lipschitz nonlinearities.

Keywords Stochastic partial differential equations · Splitting schemes · Allen–Cahn equation · Weak convergence · Kolmogorov equation

Mathematics Subject Classification 60H15 · 65C30 · 60H35

1 Introduction

In this article, we study numerical schemes introduced in by the authors in [8], for the temporal discretization of the one-dimensional stochastic Allen–Cahn equation,

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$$\frac{\partial X(t,\xi)}{\partial t} = \frac{\partial^2 X(t,\xi)}{\partial \xi^2} + X(t,\xi) - X(t,\xi)^3 + \dot{W}(t,\xi), \quad t \geq 0,\ \xi \in (0,1),$$

driven by Gaussian space-time white noise, with homogeneous Dirichlet boundary conditions.

This Stochastic partial differential equation (SPDE) has been introduced in [1] as a model for a two-phase system driven by the Ginzburg–Landau energy

$$\mathscr{E}(X) = \int |\nabla X|^2 + V(X),$$

where X is the ratio of the two species densities, and $V(X) = (X^2-1)^2$ is a double well potential. The first term in the energy models the diffusion of the interface between the two pure phases, and the second one pushes the solution to two possible stable states ± 1 (named the pure phases, i.e. minima of V). The stochastic version of the Allen–Cahn equation models the effect of thermal perturbations by an additional noise term.

The objective of this article is to study weak rates of convergence for two examples of splitting schemes schemes introduced in [8]. Let the SPDE be rewritten in the framework of [12], with $X(t) = X(t,\cdot)$:

$$dX(t) = AX(t)dt + \big(X(t) - X(t)^3\big)dt + dW(t),$$

where $\big(W(t)\big)_{t\geq 0}$ is a cylindrical Wiener process. If $\Delta t > 0$ denotes the time-step size of the integrator, the numerical schemes are defined as

$$\begin{aligned} X_{n+1}^{\rm exp} &= e^{\Delta t A}\Phi_{\Delta t}(X_n^{\rm exp}) + \int_{n\Delta t}^{(n+1)\Delta t} e^{((n+1)\Delta t - t)A} dW(t), \\ X_{n+1}^{\rm imp} &= S_{\Delta t}\Phi_{\Delta t}(X_n^{\rm imp}) + S_{\Delta t}\big(W((n+1)\Delta t) - W(n\Delta t)\big). \end{aligned} \tag{1.1}$$

In both schemes, $\big(\Phi_t\big)_{t\geq 0}$ is the flow map associated with the ODE $\dot{z} = z - z^3$, which is known exactly, see (2.5), and at each time step, one solves first the equation

$$dX(t) = \big(X(t) - X(t)^3\big)dt,$$

and second compute an approximation for the stochastic equation

$$dX(t) = AX(t)dt + dW(t).$$

In the first scheme in (1.1), the second step is also solved exactly, using an exponential integrator and exact sampling of the stochastic integral. In the second scheme, the second step is solved using a linear implicit Euler scheme, with $S_{\Delta t} = (I - \Delta t A)^{-1}$.

The schemes given by (1.1) have been introduced by the authors in [8], where two preliminary results were established: the existence of moment estimates for the numerical solution, with bounds uniform in the time-step size parameter Δt, and the mean-square convergence of the scheme, with no order of convergence. In [6], it was established that the first scheme in (1.1) has strong order of convergence $\frac{1}{4}$, based on a nice decomposition of the error. The contribution of the present article is to prove that both schemes in (1.1) have a weak order of convergence $\frac{1}{2}$.

Numerical schemes for SPDEs have been extensively studied in the last two decades, see for instance the monographs [17,22,25]. We recall that two notions of convergence are usually studied: strong convergence refers to convergence in mean-square sense, whereas weak convergence refers to convergence of distributions. If sufficiently regular test functions are considered, it is usually the case that the weak order of convergence is twice the strong order. For one-dimensional parabolic semilinear SPDEs driven by space-time white noise, the solutions are only Hölder continuous in time with exponent $\alpha < 1/4$, hence one expects a strong order of convergence equal to $\frac{1}{4}$ and a weak order equal to $\frac{1}{2}$.

In the case of globally Lipschitz continuous nonlinearities, this result has been proved in recent years for a variety of numerical schemes, exploiting different strategies for the analysis of the weak error: analysis of the error using the associated Kolmogorov equation, see [3,7,13,14,33], using the mild Itô formula approach, see [10,16,18], or other techniques, see [2,31]. In the case of SPDEs with non globally Lipschitz continuous nonlinearities, standard integrators, which treat explictly the nonlinearity, cannot be used. Design and analysis of appropriate schemes, is an active field of research, in particular concerning the stochastic Allen–Cahn equation, see [4,6,8,20,21,23,24,26,32]. Several strategies may be employed: splitting, taming, split-step methods have been proposed and studied in those references. Only strong convergence results have been obtained so far. Up to our knowledge, only the preliminary results in the PhD Thesis [19] deal with the analysis of the weak error, for split-step methods using an implicit discretization of the nonlinearity. Hence, the present article is to present detailed analysis of the weak error for a class of SPDEs with non globally Lipschitz continuous nonlinearity.

The main results of this article are Theorems 3.2 and 3.3, which may be stated as follows: for sufficiently smooth test functions φ (see Assumption 3.1), and all $\alpha \in [0, \frac{1}{2})$, $T \in (0, \infty)$,

$$\left|\mathbb{E}\big[\varphi(X(N\Delta t))\big] - \mathbb{E}\big[\varphi(X_N)\big]\right| \leq C_\alpha(T)\Delta t^\alpha.$$

In the case of the second scheme in (1.1), numerical experiments were reported in [8] to motivate and illustrate this convergence result.

As in the case of SPDEs with globally Lipschitz continuous nonlinearities, driven by a cylindrical Wiener process, the order of convergence is $\frac{1}{2}$, whereas the strong order of convergence is only $\frac{1}{4}$ in general. However, several points in the analysis are original and need to be emphasized. First, moment estimates for the numerical solution, which are non trivial in the case of numerical schemes for equations with non globally Lipschitz nonlinearities, are required. They were proved by the authors in [8] for (1.1). The analysis of the error is based on the decomposition of the error using the solution of related Kolmogorov equations, see (2.11) below and Sect. 3 for a description of the method. Appropriate regularity properties need to be proved for the first and second order Fréchet derivatives: this is presented in Theorems 4.1 and 4.2, which are auxiliary results in the analysis, but may have a more general interest. The proofs of our main results use different strategies. The proof of Theorem 3.2 is shorter, due to the use of a nice auxiliary continuous-time process, and of appropriate temporal and spatial regularity properties of the process, following [31,32]. The proof of Theorem 3.3 follows essentially the same steps as in [13] and related references, in particular a duality formula from Malliavin calculus is used. The estimate of the Malliavin derivative given in Lemma 6.4 uses in an essential way the structure as a splitting scheme, and is one of the original results used in this work.

This article is organized as follows. Assumptions, the framework and numerical schemes are given in Sect. 2. Our main results, Theorems 3.2 and 3.3 are stated in Sect. 3. The regularity properties for solutions of Kolmogorov equations are stated and proved in Sect. 4. Section 5 is devoted to the proof of Theorem 3.2, whereas Sect. 6 is devoted to the proof of Theorem 3.3.

2 Setting

We work in the standard framework of stochastic evolution equations with values in infinite dimensional separable Hilbert and Banach spaces. We refer for instance to [9,12] for details. Let $H = L^2(0,1)$, and $E = \mathscr{C}([0,1])$. We use the following notation for inner product and norms: for $x_1, x_2 \in H$, $x \in E$,

$$\langle x_1, x_2 \rangle = \int_0^1 x_1(\xi) x_2(\xi) d\xi, \quad |x_1|_H = \langle x_1, x_1 \rangle^{\frac{1}{2}}, \quad |x|_E = \max_{\xi \in [0,1]} |x(\xi)|.$$

For $p \in [1, \infty]$, we also use the notation $L^p = L^p(0,1)$ and $|\cdot|_{L^p}$ for the associated norm.

2.1 Assumptions

2.1.1 Linear operator

Let A denote the unbounded linear operator on H, with

$$\begin{cases} D(A) = H^2(0,1) \cap H_0^1(0,1), \\ Ax = \partial_\xi^2 x, \ \forall\, x \in D(A). \end{cases}$$

Let $e_n = \sqrt{2} \sin(n\pi \cdot)$ and $\lambda_n = n^2\pi^2$, for $n \in \mathbb{N}$. Note that $Ae_n = -\lambda_n e_n$, and that $\left(e_n\right)_{n \in \mathbb{N}}$ is a complete orthonormal system of H. In addition, for all $n \in \mathbb{N}$, $|e_n|_E \le \sqrt{2}$.

The linear operator A generates an analytic semi-group $\left(e^{tA}\right)_{t \ge 0}$, on L^p for $p \in [2, \infty)$ and on E. For $\alpha \in (0,1)$, the linear operators $(-A)^{-\alpha}$ and $(-A)^{\alpha}$ are constructed in a standard way, see for instance [28]:

$$(-A)^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{-\alpha}(tI - A)^{-1}dt,$$
$$(-A)^{\alpha} = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{\alpha-1}(-A)(tI - A)^{-1}dt,$$

where $(-A)^\alpha$ is defined as an unbounded linear operator on L^p. In the case $p = 2$, note that

$$(-A)^{-\alpha}x = \sum_{n\in\mathbb{N}} \lambda_n^{-\alpha}\langle x, e_n\rangle e_n, \quad x \in H,$$
$$(-A)^{\alpha}x = \sum_{n\in\mathbb{N}} \lambda_n^{\alpha}\langle x, e_n\rangle e_n,\ x \in D_2\big((-A)^\alpha\big) = \left\{x \in H; \sum_{n=1}^{\infty} \lambda_n^{2\alpha}\langle x, e_n\rangle^2 < \infty\right\}.$$

Recall the following regularity estimate: for all $\alpha \in [0, 1]$,

$$\sup_{t>0}\ \sup_{h\in H, |h|_H=1} t^\alpha |(-A)^\alpha e^{tA}h| < \infty. \tag{2.1}$$

We denote by $\mathscr{L}(H)$ the space of bounded linear operators from H to H, with associated norm denoted by $\|\cdot\|_{\mathscr{L}(H)}$. The space of Hilbert–Schmidt operators on H is denoted by $\mathscr{L}_2(H)$, and the associated norm is denoted by $\|\cdot\|_{\mathscr{L}_2(H)}$.

To conclude this section, we state several useful functional inequalities. Inequality (2.2) is a consequence of the Sobolev embedding $W^{2\eta,2}(0,1) \subset \mathscr{C}([0,1])$ when $2\eta > \frac{1}{2}$, and of the equivalence of the norms $W^{2\eta,2}(0,1)$ and $|(-A)^\eta \cdot|_{L^2}$ when $\eta \in [0, \frac{1}{2}]\backslash\{\frac{1}{4}\}$ (see for instance [15, Theorem 8.1]). For inequalities (2.3) and (2.4), we refer to [29] for the general theory, and to the arguments detailed in [7].

- For every $\eta > \frac{1}{4}$, there exists $C_\eta \in (0,\infty)$ such that

$$|(-A)^{-\eta}\cdot|_{L^2} \le C_\eta |\cdot|_{L^1}. \tag{2.2}$$

- For every $\alpha \in (0, \frac{1}{2})$, $\varepsilon > 0$, with $\alpha + \varepsilon < \frac{1}{2}$, there exists $C_{\alpha,\varepsilon} \in (0,\infty)$ such that

$$|(-A)^{-\alpha-\varepsilon}(xy)|_{L^1} \le C_{\alpha,\varepsilon}|(-A)^{\alpha+\varepsilon}x|_{L^2}|(-A)^{-\alpha}y|_{L^2}. \tag{2.3}$$

- For every $\alpha \in (0, \frac{1}{2})$, $\varepsilon > 0$, with $\alpha + 2\varepsilon < \frac{1}{2}$, there exists $C_{\alpha,\varepsilon} \in (0,\infty)$ such that, if $\psi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous,

$$|(-A)^{\alpha+\varepsilon}\psi(\cdot)|_{L^2} \le C_{\alpha,\varepsilon}[\psi]_{\mathrm{Lip}}|(-A)^{\alpha+2\varepsilon}\cdot|_{L^2}, \tag{2.4}$$

  where $[\psi]_{\mathrm{Lip}} = \sup_{z_1\neq z_2} \frac{|\psi(z_2)-\psi(z_1)|}{|z_2-z_1|}$.

These inequalities are used in an essential way to prove Lemma 2.2 below. In addition, inequality (2.2) is also used in the proof of Theorem 4.2.

2.1.2 Wiener process

Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ denote a probability space, and consider a family $\big(\beta_n\big)_{n\in\mathbb{N}}$ of independent standard real-valued Wiener processes. Then set

$$W(t) = \sum_{n\in\mathbb{N}} \beta_n(t) e_n.$$

This series does not converge in H. However, if $\tilde{H}$ is an Hilbert space, and $L \in \mathcal{L}_2(H, \tilde{H})$ is a linear, Hilbert–Schmidt, operator, then $LW(t)$ is a Wiener process on $\tilde{H}$, centered and with covariance operator $LL^\star$.

2.1.3 Nonlinearities

For all $t \geq 0$ and all $z \in \mathbb{R}$, define

$$\Phi_t(z) = \frac{z}{\sqrt{z^2 + (1-z^2)e^{-2t}}}, \quad \Psi_t(z) = \begin{cases} \frac{\Phi_t(z)-z}{t}, & t > 0, \\ z - z^3, & t = 0. \end{cases} \tag{2.5}$$

Note that for all $t \geq 0$, $z \in \mathbb{R}$, $\Phi_t(z) = z + t\Psi_t(z)$. Moreover, $\big(\Phi_t(\cdot)\big)_{t\geq 0}$ is the flow map associated with the ODE $\dot{z} = z - z^3 = \Psi_0(z)$.

Lemma 2.1 below states the properties of $\Phi_{\Delta t}$ and $\Psi_{\Delta t}$, and their derivatives, which are used in order to prove well-posedness and moment estimates, and to derive error estimates.

We refer to [8] for a detailed proof (except for the inequality concerning the second order derivative, which is not considered there but is obtained using similar arguments).

Lemma 2.1 *For every $\Delta t_0 \in (0, 1]$, there exists $C(\Delta t_0) \in (0, \infty)$ such that for all $\Delta t \in [0, \Delta t_0]$, and all $z \in \mathbb{R}$,*

$$\begin{aligned} \big|\Phi'_{\Delta t}(z)\big| &\leq e^{\Delta t_0}, & \Psi'_{\Delta t}(z) &\leq e^{\Delta t_0}, \\ \big|\Psi'_{\Delta t}(z)\big| &\leq C(\Delta t_0)(1 + |z|^2), & \big|\Psi''_{\Delta t}(z)\big| &\leq C(\Delta t_0)(1 + |z|), \\ \big|\Psi_{\Delta t}(z) - \Psi_0(z)\big| &\leq C(\Delta t_0)\Delta t(1 + |z|^5). \end{aligned}$$

In particular, the mapping $\Psi_{\Delta t}$ satisfies the following one-sided Lipschitz condition: for all $z_1, z_2 \in \mathbb{R}$,

$$\big(\Psi_{\Delta t}(z_2) - \Psi_{\Delta t}(z_1)\big)\big(z_2 - z_1\big) \leq e^{\Delta t}|z_2 - z_1|^2.$$

Observe also that for $\Delta t > 0$, the mapping $\Psi_{\Delta t}$ is of class $\mathcal{C}^\infty$, and admits bounded first and second order derivatives. However, such bounds are not uniform with respect to $\Delta t > 0$.

We conclude this section with an auxiliary result, see [32] for a similar statement.

Lemma 2.2 *For every $\Delta t_0 \in (0,1]$, $\eta \in (\frac{1}{4},1)$, $\alpha \in (0,\frac{1}{2})$, and $\varepsilon > 0$ such that $\alpha + 2\varepsilon < \frac{1}{2}$, there exists $C(\Delta t_0,\eta,\alpha,\varepsilon) \in (0,\infty)$ such that for all $\Delta t \in [0,\Delta t_0]$ for all $x, y \in H$, with $|(-A)^{\frac{\alpha}{2}+\varepsilon}x|_H < \infty$,*

$$|(-A)^{-\eta-\frac{\alpha+\varepsilon}{2}}\big(\Psi'_{\Delta t}(x)y\big)|_H \leq C(\Delta t_0,\eta,\alpha,\varepsilon)(1+|x|_E)|(-A)^{\frac{\alpha}{2}+\varepsilon}x|_H|(-A)^{-\frac{\alpha}{2}}y|_H.$$

Proof Using successively the inequalities (2.2) and (2.3),

$$\begin{aligned}|(-A)^{-\eta-\frac{\alpha+\varepsilon}{2}}\big(\Psi'_{\Delta t}(x)y\big)|_{L^2} &\leq C_\eta|(-A)^{-\frac{\alpha+\varepsilon}{2}}\big(\Psi'_{\Delta t}(x)y\big)|_{L^1}\\ &\leq C_{\eta,\alpha,\varepsilon}|(-A)^{\frac{\alpha+\varepsilon}{2}}\Psi'_{\Delta t}(x)|_{L^2}|(-A)^{-\frac{\alpha}{2}}y|_{L^2}.\end{aligned}$$

Since the function $\Psi'_{\Delta t}$ is not globally Lipschitz continuous when $\Delta t = 0$, and we have $\lim_{\Delta t\to 0}\ [\Psi'_{\Delta t}]_{\mathrm{Lip}} = \infty$, the inequality (2.4) cannot be used directly.

However, the derivative $\Psi''_{\Delta t}$ has at most linear growth, uniformly in $\Delta t \in [0,\Delta t_0]$: $|\Psi''_{\Delta t}(z)| \leq C(\Delta t_0)(1+|z|)$ for all $z \in \mathbb{R}$.

To proceed, a truncation argument is employed. Let $x \in E$ and let $R = 1 + |x|_E$. Then one can construct a smooth function $\theta^R_{\Delta t} : \mathbb{R} \to \mathbb{R}$, such that $\theta^R_{\Delta t}$ coincides with $\Psi'_{\Delta t}$ on $[-R,R]$, and $[\theta^R_{\Delta t}]_{\mathrm{Lip}} \leq C(\Delta t_0)(1+R)$. Then by construction one has $\Psi'_{\Delta t}(x) = \theta^R_{\Delta t}(x)$, and applying inequality (2.4) with $\theta^R_{\Delta t}$ implies

$$|(-A)^{\frac{\alpha+\varepsilon}{2}}\Psi'_{\Delta t}(x)|_{L^2} = |(-A)^{\frac{\alpha+\varepsilon}{2}}\theta^R_{\Delta t}(x)|_{L^2} \leq C_{\alpha,\varepsilon}C(\Delta t_0)(1+R)|(-A)^{\frac{\alpha+2\varepsilon}{2}}x|_{L^2},$$

with $R = |x|_E + 1$. This concludes the proof of Lemma 2.2. □

2.2 Stochastic partial differential equations

The stochastic Allen–Cahn equation with additive space-time white noise perturbation, is

$$dX(t) = AX(t)dt + \big(X(t) - X(t)^3\big)dt + dW(t), \quad X(0) = x. \tag{2.6}$$

More generally, for $\Delta t \in [0,1]$ introduce the auxiliary equation

$$dX^{(\Delta t)} = AX^{(\Delta t)}(t)dt + \Psi_{\Delta t}(X^{(\Delta t)}(t))dt + dW(t), \quad X^{(\Delta t)} = x. \tag{2.7}$$

If $\Delta t > 0$, since $\Psi_{\Delta t}$ is globally Lipschitz continuous, standard fixed point arguments (see for instance [12, Chapter 7, Theorem 7.5]) imply that, for any initial condition $x \in H$, the SPDE (2.7) admits a unique global mild solution $\big(X^{(\Delta t)}(t,x)\big)_{t\geq 0}$, *i.e.* which satisfies

$$X^{(\Delta t)}(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}\Psi_{\Delta t}(X^{(\Delta t)}(s,x))ds + \int_0^t e^{(t-s)A}dW(s), \quad t \geq 0.$$

If $\Delta t = 0$, proving global well-posedness requires more refined arguments, in particular the use of the one-sided Lipschitz condition (see for instance [9]). For any initial condition $x \in E$, there exists a unique mild solution $\big(X(t,x)\big)_{t\geq 0}$ of Eq. (2.6), and $X^{(0)} = X$ solves Eq. (2.7) with $\Delta t = 0$:

$$X^{(0)}(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}\Psi_0(X^{(0)}(s,x))ds + \int_0^t e^{(t-s)A}dW(s), \quad t \geq 0.$$

To simplify notation, we often write $X(t)$ and $X^{(\Delta t)}(t)$ and omit the initial condition x.

Let

$$W^A(t) = \int_0^t e^{(t-s)A}dW(s). \tag{2.8}$$

Then (see [9, Lemma 6.1.2]) for every $T \in (0,\infty)$ and $M \in \mathbb{N}$, there exists $C(T,M) \in (0,\infty)$ such that

$$\mathbb{E}[\sup_{t\in[0,T]} |W^A(t)|_E^M] \leq C(T,M). \tag{2.9}$$

Combined with the one-sided Lipschitz condition for $\Psi_{\Delta t}$, see Lemma 2.1, (2.9) yields moment estimates for $X^{(\Delta t)}$.

Lemma 2.3 *Let $T \in (0,\infty)$, $\Delta t_0 \in (0,1]$ and $M \in \mathbb{N}$. There exists $C(T,\Delta t_0,M) \in (0,\infty)$ such that, for all $\Delta t \in [0,\Delta t_0]$ and $x \in E$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|X^{(\Delta t)}(t,x)\right|_E^M\right] \leq C(T,\Delta t_0,M)(1+|x|_E^M).$$

We refer to [8] for a proof.

2.3 Kolmogorov equations

In this section, we introduce functions $u^{(\Delta t)}$ which play a key role in the weak error analysis, and which are solutions of infinite dimensional Kolmogorov equations associated with (2.7).

Let $\varphi : H \to \mathbb{R}$ be a twice Fréchet differentiable function, with bounded first and second order Fréchet derivatives, denoted by $D\varphi(x)$ and $D^2\varphi(x)$ respectively, for all $x \in H$. Owing to the Riesz Theorem, $D\varphi(x)$, resp. $D^2\varphi(x)$, is identified with an element of H, resp. of $\mathscr{L}(H)$.

Let $\Delta t \in (0,1]$. Note that we do not consider the case $\Delta t = 0$ in this section, the reason why will be clear below. For every $t \geq 0$, set

$$u^{(\Delta t)}(t,x) = \mathbb{E}\big[\varphi(X^{(\Delta t)}(t,x))\big]. \tag{2.10}$$

Formally, $u^{(\Delta t)}$ is solution of the Kolmogorov equation associated with (2.7):

$$\begin{aligned}\frac{\partial u^{(\Delta t)}(t,x)}{\partial t} &= \mathscr{L}^{(\Delta t)}u^{(\Delta t)}(t,x)\\ &= \langle Ax + \Psi_{\Delta t}(x), Du^{(\Delta t)}(t,x)\rangle + \frac{1}{2}\sum_{j\in\mathbb{N}} D^2u^{(\Delta t)}(t,x).(e_j,e_j),\end{aligned} \tag{2.11}$$

where D and D^2 stand for the first and second order Fréchet derivative with respect to the x variable, respectively. We refer to [9] for a general analysis of infinite dimensional Kolmogorov equations, see also the recent work [7] and references therein.

A rigorous meaning for the infinite dimensional partial differential equation (2.11) can be given using an appropriate regularization procedure (which is only used in theoretical analysis, and is not required in practical implementation of the scheme). Since $\Psi_{\Delta t}$ is globally Lipschitz continuous for fixed $\Delta t > 0$, this may be performed by a standard spectral Galerkin approximation. This is the strategy which is used in most of the works in the literature, see for instance [13]: in the finite dimensional approximation, all linear operators are bounded and sums are finite. However, choosing this spectral Galerkin method would not allow us to pass to the limit $\Delta t \to 0$ in estimates and keep bounds uniform in the regularization parameter. Instead, following [7], we propose to replace the noise $dW(t)$ in (2.7) by $e^{\delta A}dW(t)$, with the regularization parameter $\delta > 0$. This modifies the sum appearing on the right-hand side of (2.11): a factor $e^{-2\lambda_j \delta}$ appears, hence the convergence of the sum just follows from boundedness of the second-order derivative. The regularization of the noise term in (2.7) also improves the spatial regularity of the process, hence a simpler treatment for the terms where the unbounded linear operator A appears.

Below, rigorous computations are performed with fixed $\delta > 0$, owing to regularization estimates for $e^{\delta A}$ (with constants depending on δ). However, in all the estimates below, note that constants independent of δ are obtained, since only the inequality $|e^{\delta A}x|_{L^p} \leq |x|_{L^p}$, for all $p \in [2, \infty]$ and $x \in L^p$, is be employed in the proofs. To simplify notation, we do not mention the regularization parameter δ in the computations and statements below. All the results may be interpreted as follows: the estimates hold true for the regularized equation (with $\delta > 0$), and since the constants in the upper bounds do not depend on δ, passing to the limit $\delta \to 0$ yields results for the SPDE driven by space-time white noise.

2.4 Splitting schemes

We are now in position to rigorously define the numerical schemes which are studied in this article.

A first scheme is defined as:

$$X_{n+1} = e^{\Delta t A}\Phi_{\Delta t}(X_n) + \int_{n\Delta t}^{(n+1)\Delta t} e^{(n\Delta t - t)A}dW(t). \tag{2.12}$$

A second scheme is defined as:

$$X_{n+1} = S_{\Delta t}\Phi_{\Delta t}(X_n) + S_{\Delta t}\big(W((n+1)\Delta t) - W(n\Delta t)\big), \tag{2.13}$$

where $S_{\Delta t} = (I - \Delta t A)^{-1}$.

The schemes are constructed following a Lie–Trotter splitting strategy (see [30]) for the SPDE (2.6). Firstly, the equation $dX(t) = \Psi_0(X(t))dt$ is solved explictly using the flow map at time $t = \Delta t$, namely $\Phi_{\Delta t}$. Secondly, the equation $dX(t) =$

$AX(t)dt + dW(t)$ is either solved exactly in scheme (2.12), or using a linear implicit Euler scheme in (2.13).

As already emphasized in [6,8], observe that (2.12) and (2.13) can be interpreted as integrators for the auxiliary equation (2.7) with nonlinear coefficient $\Psi_{\Delta t}$: respectively

$$X_{n+1} = e^{\Delta t A} X_n + \Delta t e^{\Delta t A} \Psi_{\Delta t}(X_n) + \int_{n\Delta t}^{(n+1)\Delta t} e^{(n\Delta t - t)A} dW(t),$$
$$X_{n+1} = S_{\Delta t} X_n + \Delta t S_{\Delta t} \Psi_{\Delta t}(X_n) + S_{\Delta t}\big(W((n+1)\Delta t) - W(n\Delta t)\big),$$

where in (2.12), an exponential Euler integrator is used, whereas in (2.13) a semi-implicit integrator is used.

Moment estimates are available. We refer to [8] for a proof.

Lemma 2.4 *Let $T \in (0, \infty)$, $\Delta t_0 \in (0, 1]$ and $M \in \mathbb{N}$. There exists $C(T, \Delta t_0, M) \in (0, \infty)$ such that, for all $\Delta t \in [0, \Delta t_0]$ and $x \in E$,*

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}, n\Delta t \le T} |X_n|_E^M\right] \le C(T, \Delta t_0, M)\left(1 + |x|_E^M\right),$$

where the sequence $\left(X_n\right)_{n\ge 0}$ defined either by (2.12) or by (2.13).

3 Weak convergence results

This section is devoted to the statement of the main result of this article: the numerical schemes (2.12) and (2.13) have a weak convergence rate equal to $\frac{1}{2}$, see Theorems 3.2 and 3.3 below respectively.

The main difficulty and novelty of this contribution is the treatment of SPDEs with non globally Lipschitz continuous nonlinear coefficient. Up to our knowledge, except in the PhD Thesis [19] (where split-step schemes based on an implicit discretization of the nonlinear term, for more general polynomial coefficients, are considered), there is no analysis of weak rates of convergence for that situation in the literature.

Strong convergence of numerical schemes (2.12) and (2.13) is proved in [8], without rate. In [6], the strong rate of convergence $\frac{1}{4}$ is proved for the scheme (2.12).

Test functions satisfy the following condition.

Assumption 3.1 The function $\varphi : H \to \mathbb{R}$ is twice Fréchet differentiable, and has bounded first and second order Fréchet derivatives:

$$\|\varphi\|_{1,\infty} = \sup_{x\in H, h\in H, |h|_H=1} |D\varphi(x).h| < \infty,$$
$$\|\varphi\|_{2,\infty} = \|\varphi\|_{1,\infty} + \sup_{x\in H, h,k\in H, |h|_H=|k|_H=1} |D^2\varphi(x).(h,k)| < \infty.$$

We are now in position to state our main results.

Theorem 3.2 *Let $(X_n)_{n\in\mathbb{N}}$ be defined by the scheme* (2.12).

Let $T\in(0,\infty)$, $\Delta t_0\in(0,1]$ and $x\in E$. For all $\alpha\in[0,\frac{1}{2})$, there exists a constant $C_\alpha(T,\Delta t_0,|x|_E)\in(0,\infty)$ such that the following holds true.

Let φ satisfy Assumption 3.1. *For all $\Delta t\in(0,\Delta t_0]$ and $N\in\mathbb{N}$, such that $T=N\Delta t$,*

$$\left|\mathbb{E}\big[\varphi(X(N\Delta t))\big]-\mathbb{E}\big[\varphi(X_N)\big]\right|\leq C_\alpha(T,\Delta t_0,|x|_E)\|\varphi\|_{1,\infty}\Delta t^\alpha, \tag{3.1}$$

Theorem 3.3 *Let $(X_n)_{n\in\mathbb{N}}$ be defined by the scheme* (2.13).

Let $T\in(0,\infty)$, $\Delta t_0\in(0,1]$ and $x\in E$. For all $\alpha\in[0,\frac{1}{2})$, there exists a constant $C_\alpha(T,\Delta t_0,|x|_E)\in(0,\infty)$ such that the following holds true.

Let φ satisfy Assumption 3.1. *For all $\Delta t\in(0,\Delta t_0]$ and $N\in\mathbb{N}$, such that $T=N\Delta t$,*

$$\left|\mathbb{E}\big[\varphi(X(N\Delta t))\big]-\mathbb{E}\big[\varphi(X_N)\big]\right|\leq C_\alpha(T,\Delta t_0,|x|_E)\|\varphi\|_{2,\infty}\Delta t^\alpha. \tag{3.2}$$

Previous Theorems 3.2 and 3.3 are natural generalizations of the results obtained for with globally Lipschitz continuous nonlinear coefficient, see for instance [13]. We obtain the same weak order of convergence $\frac{1}{2}$, which is twice the strong order of convergence. Up to our knowledge, this is the first contribution in the literature proving that this result holds true for parabolic semilinear SPDEs with non-globally Lipschitz continuous nonlinear coefficient.

Numerical experiments which illustrate Theorem 3.3 are reported in [8].

Remark 3.4 The regularity of the function φ is essential to get a weak order of convergence which is twice the strong order, as proved in [5]. If one wants to replace $\|\varphi\|_{2,\infty}$ by $\|\varphi\|_{1,\infty}$ in the right-hand side of (3.2), the order of convergence has to be replaced by $\frac{\alpha}{2}$, even in the absence of nonlinear coefficient.

In the right-hand side of (3.1), it is sufficient to control only $\|\varphi\|_{1,\infty}$. This is due to an appropriate decomposition of the weak error. This is not in contradiction with [5]: in the absence of nonlinear coefficient, the weak error is equal to zero.

Proving Theorems 3.2 and 3.3 is the aim of the remainder of the article. The strategy consists in decomposing the weak error as follows:

$$\begin{aligned}\left|\mathbb{E}\big[\varphi(X(N\Delta t))\big]-\mathbb{E}\big[\varphi(X_N)\big]\right|&\leq\left|\mathbb{E}\big[\varphi(X(N\Delta t))\big]-\mathbb{E}\big[\varphi(X^{(\Delta t)}(N\Delta t))\big]\right|\\&\quad+\left|\mathbb{E}\big[\varphi(X^{(\Delta t)}(N\Delta t))\big]-\mathbb{E}\big[\varphi(X_N)\big]\right|.\end{aligned}$$

The first error term is estimated using the following result, quoted from [8], combined with globally Lipschitz continuity of φ induced by Assumption 3.1.

Proposition 3.5 *Let $T\in(0,\infty)$, $\Delta t_0\in(0,1]$ and $x\in E$. There exists a positive real number $C(T,\Delta t_0,|x|_E)\in(0,\infty)$ such that for all $\Delta t\in(0,\Delta t_0]$,*

$$\sup_{t\in[0,T]}\mathbb{E}\big[\big|X(t)-X^{(\Delta t)}(t))\big|_H\big]\leq C(T,\Delta t_0,|x|_E)\Delta t.$$

The treatment of the second error term requires more subtle arguments. First, thanks to (2.10), and a telescoping sum argument,

$$\begin{aligned}\mathbb{E}\big[\varphi(X^{(\Delta t)}(n\Delta t))\big]-\mathbb{E}\big[\varphi(X_n)\big]&=u^{(\Delta t)}(n\Delta t,x)-\mathbb{E}\big[u^{(\Delta t)}(0,X_n)\big]\\&=\sum_{k=0}^{n-1}\Big(\mathbb{E}\big[u^{(\Delta t)}((n-k)\Delta t,X_k)\\&\quad-u^{(\Delta t)}((n-k-1)\Delta t,X_{k+1})\big]\Big).\end{aligned}$$

The details then depend on the numerical scheme. First, an auxiliary continuous-time process $\tilde{X}$ is introduced, see (5.1) and (6.1). It satisfies $\tilde{X}(k\Delta t)=X_k$ for all $k\in\mathbb{N}$. Second, Itô's formula is applied, and the Kolmogorov equation (2.11) is used. Theorem 3.2 [numerical scheme given by (2.12)] is proved in Sect. 5. Theorem 3.3 [numerical scheme given by (2.13)] is proved in Sect. 6.

Spatial derivatives $Du^{(\Delta t)}(t,x)$ and $D^2u^{(\Delta t)}(t,x)$ appear in the expansion of the error obtained following this standard strategy. In infinite dimension, see [3,7,13,33], appropriate regularity properties are required to obtain the weak order of convergence $\frac{1}{2}$. They are studied in Sect. 4. Recall that computations reported in the proofs below are justified using the regularization procedure mentioned above ($\delta>0$), however it is important to note that the constants do not depend on δ.

4 Regularity properties for solutions of Kolmogorov equations

This section is devoted to state and prove regularity properties of the function $u^{(\Delta t)}$, defined by (2.10), solution of the Kolmogorov equation (2.11) associated to the auxiliary equation (2.7). The main difficulty and novelty is due to the poor regularity property of $\Psi_{\Delta t}$: even if for fixed Δt, $\Psi_{\Delta t}$ is globally Lipschitz continuous, there is no bound which is uniform in $\Delta t>0$, since Ψ_0 is polynomial of degree 3.

Theorems 4.1 and 4.2 below are the main results of this section, and they are of interest beyond the analysis of weak convergence rates. They are natural generalizations in a non-globally Lipschitz framework of the estimates provided in [13], and extended in [7] with nonlinear diffusion coefficients. We emphasize that the right-hand sides in the estimates (4.1) and (4.2) do not depend on Δt.

Theorem 4.1 *Let $T\in(0,\infty)$ and $\Delta t_0\in(0,1]$. For every $\alpha\in[0,1)$, there exists a constant $C_\alpha(T,\Delta t_0)\in(0,\infty)$ such that, for all $\Delta t\in(0,\Delta t_0]$, $x\in E$, $h\in H$ and $t\in(0,T]$,*

$$|Du^{(\Delta t)}(t,x).h|\leq\frac{C_\alpha(T,\Delta t_0)(1+|x|_E^2)\|\varphi\|_{1,\infty}}{t^\alpha}|(-A)^{-\alpha}h|_H. \tag{4.1}$$

This theorem may be interpreted as a regularization property: the assumption $Du^{(\Delta t)}(0,x)\in H$ implies that, for positive t, $(-A)^\alpha Du^{(\Delta t)}(t,x)\in H$ with $\alpha\in(0,1)$. This is related to the estimate (2.1) for the semi-group generated by A.

Theorem 4.2 *Let $T \in (0, \infty)$ and $\Delta t_0 \in (0, 1]$. For every $\beta, \gamma \in [0, 1)$, with the condition $\beta + \gamma < 1$, there exists $C_{\beta,\gamma}(T, \Delta t_0) \in (0, \infty)$ such that, for all $\Delta t \in (0, \Delta t_0]$, $x \in E$, $h, k \in H$ and $t \in (0, T]$,*

$$|D^2 u^{(\Delta t)}(t, x).(h, k)| \le \frac{C_{\beta,\gamma}(T, \Delta t_0)(1 + |x|_E^7)\|\varphi\|_{2,\infty}}{t^{\beta+\gamma}} |(-A)^{-\beta} h|_H |(-A)^{-\gamma} k|_H. \tag{4.2}$$

Let us recall that the existence of the derivatives in Theorems 4.1 and 4.2 is justified by the regularization procedure (with $\delta > 0$) mentioned above. Note that the upper bounds indeed do not depend on δ.

Remark 4.3 Results similar to Theorems 4.1 and 4.2 are studied in [19], with different techniques.

Note that we obtain stronger results. In Theorem 4.1, one may choose $\alpha \in [0, 1)$ instead of $\alpha \in [0, \frac{1}{2})$. In Theorem 4.2, one may choose $\beta, \gamma \in [0, 1)$ satisfying the condition $\beta + \gamma < 1$, instead of $\beta, \gamma \in [0, \frac{1}{2})$. The stronger statements are useful below, to simplify the treatments of several error terms.

Expressions of $Du^{(\Delta t)}(t, x)$ and $D^2 u^{(\Delta t)}(t, x)$ are given below: for $h, k \in H$, $x \in H$, and $t \ge 0$,

$$\begin{aligned} Du^{(\Delta t)}(t, x).h &= \mathbb{E}\big[D\varphi(X^{(\Delta t)}(t, x)).\eta^h(t, x)\big], \\ D^2 u^{(\Delta t)}(t, x).(h, k) &= \mathbb{E}\big[D\varphi(X^{(\Delta t)}(t, x)).\zeta^{h,k}(t, x)\big] \\ &\quad + \mathbb{E}\big[D^2\varphi(X^{(\Delta t)}(t, x)).(\eta^h(t, x), \eta^k(t, x))\big], \end{aligned} \tag{4.3}$$

where the processes $\eta^h(\cdot, x)$ and $\zeta^{h,k}(\cdot, x)$ are the solutions the random PDEs

$$\frac{d\eta^h(t, x)}{dt} = A\eta^h(t, x) + \Psi'_{\Delta t}(X^{(\Delta t)}(t, x))\eta^h(t, x), \tag{4.4}$$

with initial condition $\eta^h(0, x) = h$, and

$$\begin{aligned} \frac{d\zeta^{h,k}(t, x)}{dt} &= A\zeta^{h,k}(t, x) + \Psi'_{\Delta t}(X^{(\Delta t)}(t, x))\zeta^{h,k}(t, x) \\ &\quad + \Psi''_{\Delta t}(X^{(\Delta t)}(t, x))\eta^h(t, x)\eta^k(t, x), \end{aligned} \tag{4.5}$$

with initial condition $\zeta^{h,k}(0, x) = 0$. The expressions above are obtained by straightforward differentiation under the expectation sign. We refer for instance to the monograph [9] for details (in more general situations).

To simplify notation, the parameter Δt is omitted in the notation for $\eta^h(t, x)$ and $\zeta^{h,k}(t, x)$.

4.1 Proof of Theorem 4.1

Following the strategy in [7], introduce the auxiliary process

$$\tilde{\eta}^h(t,x) = \eta^h(t,x) - e^{tA}h.$$

Then, thanks to (4.3),

$$Du^{(\Delta t)}(t,x).h = \mathbb{E}\big[D\varphi(X^{(\Delta t)}(t,x)).(e^{tA}h)\big] + \mathbb{E}\big[D\varphi(X^{(\Delta t)}(t,x)).\tilde{\eta}^h(t,x)\big],$$

and thanks to (4.4),

$$\begin{aligned}\frac{d\tilde{\eta}^h(t,x)}{dt} &= A\tilde{\eta}^h(t,x) + \Psi'_{\Delta t}(X^{(\Delta t)}(t,x))\tilde{\eta}^h(t,x)\\ &\quad + \Psi'_{\Delta t}(X^{(\Delta t)}(t,x))e^{tA}h, \quad \tilde{\eta}^h(0) = 0.\end{aligned}$$

On the one hand, for any $\alpha \in [0,1)$, $h \in H$ and $t > 0$, using (2.1), one has

$$\big|\mathbb{E}\big[D\varphi(X^{(\Delta t)}(t,x)).(e^{tA}h)\big]\big| \le \|\varphi\|_{1,\infty}|e^{tA}h|_H \le \frac{C_\alpha}{t^\alpha}\|\varphi\|_{1,\infty}|(-A)^{-\alpha}h|_H. \quad (4.6)$$

On the other hand, the process $\tilde{\eta}^h$ may be expressed as

$$\tilde{\eta}^h(t,x) = \int_0^t U(t,s)\big(\Psi'_{\Delta t}(X^{(\Delta t)}(s,x))e^{sA}h\big)ds, \quad (4.7)$$

where $\big(U(t,s)h\big)_{t\ge s}$ solves, for every $h \in H$,

$$\frac{dU(t,s)h}{dt} = \big(A + \Psi'_{\Delta t}(X^{(\Delta t)}(t,x))\big)U(t,s)h, \quad U(s,s)h = h. \quad (4.8)$$

A straightforward energy estimate, using the one-sided Lipschitz condition for $\Psi'_{\Delta t}$, yields

$$\frac{1}{2}\frac{d|U(t,s)h|_H^2}{dt} \le (e^{\Delta t_0} - \lambda_1)|U(t,s)h|_H^2,$$

thus by Gronwall's Lemma $|U(t,s)h|_H^2 \le C(T,\Delta t_0)|h|_H^2$, for all $s \le t \le T$.

Thus, thanks to (4.7), for $\alpha \in [0,1)$,

$$\begin{aligned}|\tilde{\eta}^h(t,x)|_H &\le C(T,\Delta t_0)\int_0^t |\Psi'_{\Delta t}(X^{(\Delta t)}(s,x))e^{sA}h|_H ds\\ &\le C(T,\Delta t_0)\sup_{s\in[0,T]}|\Psi'_{\Delta t}(X^{(\Delta t)}(s,x))|_E\int_0^t \frac{C_\alpha}{s^\alpha}ds|(-A)^{-\alpha}h|_H.\end{aligned}$$

Thanks to Lemmas 2.1 and 2.3,

$$\left|\mathbb{E}\left[D\varphi(X^{(\Delta t)}(t,x)).\tilde{\eta}^h(t,x)\right]\right| \le \frac{C_\alpha(T,\Delta t_0)}{t^\alpha}\left(1+|x|_E^2\right)\|\varphi\|_{1,\infty}|(-A)^{-\alpha}h|_H. \quad (4.9)$$

Combining (4.6) and (4.9) concludes the proof of (4.1).

For future reference, note that for $t \in (0, T]$,

$$|\eta^h(t,x)|_H \le \frac{C_\alpha(T,\Delta t_0)}{t^\alpha}|(-A)^{-\alpha}h|_H\left(1+\sup_{s\in[0,T]}|\Psi'_{\Delta t}(X^{(\Delta t)}(s,x))|_E\right). \quad (4.10)$$

4.2 Proof of Theorem 4.2

Thanks to (4.3),

$$|D^2u^{(\Delta t)}(t,x).(h,k)| \le \|\varphi\|_{1,\infty}\mathbb{E}[|\zeta^{h,k}(t,x)|_H]+\|\varphi\|_{2,\infty}\mathbb{E}[|\eta^h(t,x)|_H|\eta^k(t,x)|_H].$$

On the one hand, thanks to (4.10), using Lemmas 2.1 and 2.3, we obtain

$$\mathbb{E}[|\eta^h(t,x)|_H|\eta^k(t,x)|_H] \le \frac{C_{\beta,\gamma}(T,\Delta t_0)(1+|x|_E^4)}{t^{\beta+\gamma}}|(-A)^{-\beta}h|_H|(-A)^{-\gamma}|_H, \quad (4.11)$$

for all $\beta,\gamma \in [0,1)$.

On the other hand, the process $\zeta^{h,k}$ which is solution of (4.5), can be expressed in terms of the operators $U(t,s)$ introduced in Eq. (4.8): one has

$$\zeta^{h,k}(t,x) = \int_0^t U(t,s)\left(\Psi''_{\Delta t}(X^{(\Delta t)}(s,x))\eta^h(s,x)\eta^k(s,x)\right)ds. \quad (4.12)$$

Thanks to (2.2),

$$\begin{aligned}&\left|(-A)^{-\frac{1}{2}}\left(\Psi''_{\Delta t}(X^{(\Delta t)}(s,x))\eta^h(s,x)\eta^k(s,x)\right)\right|_H\\ &\quad\le C\left|\Psi''_{\Delta t}(X^{(\Delta t)}(s,x))\right|_E|\eta^h(s,x)|_H|\eta^k(s,x)|_H.\end{aligned}$$

The following result allows us to use this inequality in (4.12).

Lemma 4.4 *Let $T \in (0,\infty)$ and $\Delta t_0 \in (0,1]$. There exists $C(T,\Delta t_0) \in (0,\infty)$ such that, for all $\Delta t \in (0,\Delta t_0]$, $x \in E$, $h \in H$, $0 \le s < t \le T$,*

$$|U(t,s)h|_H \le \frac{C(T,\Delta t_0)}{(t-s)^{\frac{1}{2}}}\left(1+\sup_{0\le r\le T}|\Psi'_{\Delta t}(X^{(\Delta t)}(r,x))|_E\right)|(-A)^{-1/2}h|_H.$$

The proof of Lemma 4.4 is postponed to Sect. 4.3. We refer to [11] for a similar result and the idea of the proof.

Thanks to Lemma 4.4 and to (4.12), we get

$$\mathbb{E}|\zeta^{h,k}(t,x)|_H \leq \int_0^t \frac{C(T,\Delta t_0)}{(t-s)^{\frac{1}{2}}} \Big(1 + \sup_{0\leq r\leq T} |\Psi'_{\Delta t}(X^{(\Delta t)}(r,x))|_E\Big) \big|\Psi''_{\Delta t}(X^{(\Delta t)}(s,x))\big|_E |\eta^h(s,x)|_H |\eta^k(s,x)|_H ds.$$

Thanks to (4.10), and Lemmas 2.1 and 2.3, we obtain

$$\begin{aligned}&\mathbb{E}|\zeta^{h,k}(t,x)|_H \\ &\leq C_{\beta,\gamma}(T,\Delta t_0)\big(1+|x|_E^7\big)\int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{\beta+\gamma}} ds\ |(-A)^\beta h|_H |(-A)^\gamma k|_H,\end{aligned}$$

hence

$$\mathbb{E}|\zeta^{h,k}(t,x)|_H \leq \frac{C_{\beta,\gamma}(T,\Delta t_0)(1+|x|_E^7)}{t^{\beta+\gamma}} |(-A)^{-\beta} h|_H |(-A)^{-\gamma} k|_H. \quad (4.13)$$

Combining (4.11) and (4.13), thanks to (4.3), then concludes the proof of (4.2).

4.3 Proof of Lemma 4.4

We claim that for all $0 \leq s \leq T$ and $h \in H$,

$$\int_s^T |U(r,s)h|_H^2 dr \leq C(T,\Delta t_0)\Big(1 + \sup_{0\leq r\leq T} |\Psi'_{\Delta t}(X^{(\Delta t)}(r,x))|_E\Big) |(-A)^{-1/2} h|_H^2. \quad (4.14)$$

Lemma 4.4 is then a straightforward consequence of (4.14), using the mild formulation

$$U(t,s)h = e^{(t-s)A}h + \int_s^t e^{(t-r)A}\Psi'_{\Delta t}(X^{(\Delta t)}(s,x))U(r,s)h ds.$$

It remains to prove (4.14). Let $s \in [0,T]$ be fixed, and define

$$\mathscr{U}_s : h \in H \mapsto (U(t,s)h)_{s\leq t\leq T} \in L^2(s,T;H).$$

Introduce $\mathscr{U}_s^\star : L^2(s,T;H) \to H$ the adjoint of $\mathscr{U}_s$. Then, by a duality argument, the claim (4.14) is a straightforward consequence of the following estimate: for all $F \in L^2(s,T;H)$,

$$|(-A)^{\frac{1}{2}}\mathscr{U}_s^\star F|^2 \leq C(T,\Delta t_0)\big(1 + \sup_{0\leq r\leq T} |\Psi'_{\Delta t}(X^{(\Delta t)}(r,x))|_E\big)\int_s^T |F(r)|_H^2 dr. \quad (4.15)$$

To prove the inequality (4.15), let $F \in L^2(s,T;H)$, and observe that $\mathscr{U}_s^\star F = \xi_s(s)$, where $\big(\xi_s(t)\big)_{s\le t\le T}$ is the solution of the backward evolution equation

$$\frac{d\xi_s(t)}{dt} = -A\xi_s(t) - \Psi'_{\Delta t}(X^{(\Delta t)}(t,x))\xi_s(t) - F(t), \quad \xi_s(T) = 0.$$

Indeed, for all $h \in H$,

$$\langle h, \mathscr{U}_s^\star(F)\rangle = \int_s^T \langle U(t,s)h, F(t)\rangle dt = -\int_s^T \frac{d}{dt}\big(\langle U(t,s)h, \xi_s(t)\rangle\big)dt = \langle h, \xi_s(s)\rangle,$$

using the conditions $\xi_s(T) = 0$ and $U(s,s)h = h$.

To obtain the required estimate of $|(-A)^{\frac12}\mathscr{U}_s^\star F|^2 = |(-A)^{\frac12}\xi_s(s)|^2$, compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|(-A)^{\frac12}\xi_s(t)|_H^2 &= |(-A)\xi_s(t)|_H^2 - \langle \Psi'_{\Delta t}(X^{(\Delta t)}(t,x))\xi_s(t), (-A)\xi_s(t)\rangle \\ &\quad - \langle F(t), (-A)\xi_s(t)\rangle \\ &\ge |(-A)\xi_s(t)|_H^2 - |\Psi'_{\Delta t}(X^{(\Delta t)}(t,x))|_E |\xi_s(t)|_H |(-A)\xi_s(t)|_H \\ &\quad - |F(t)|_H |(-A)\xi_s(t)|_H \\ &\ge -\frac{1}{2}|\Psi'_{\Delta t}(X^{(\Delta t)}(t,x))|_E^2 |\xi_s(t)|_H^2 - \frac{1}{2}|F(t)|_H^2,\end{aligned}$$

thanks to Young's inequality. Integrating from $t = s$ to $t = T$, and using $\xi_s(T) = 0$, we have

$$|(-A)^{\frac12}\xi_s(s)|^2 \le \sup_{0\le t\le T} |\Psi'_{\Delta t}(X(t,x))|_E^2 \int_s^T |\xi_s(t)|^2 dt + \frac{1}{2}\int_s^T |F(t)|_H^2 dt. \tag{4.16}$$

Moreover, using $-\Psi'_{\Delta t}(\cdot) \ge -e^{\Delta t_0}$, see Lemma 2.1, thanks to Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\xi_s(t)|_H^2 &= |(-A)^{1/2}\xi_s(t)|_H^2 - e^{\Delta t_0}|\xi_s(t)|_H^2 - \langle F(t), \xi_s(t)\rangle \\ &\ge |(-A)^{1/2}\xi_s(t)|_H^2 - e^{\Delta t_0}|\xi_s(t)|_H^2 - |(-A)^{-\frac12}F(t)|_H |(-A)^{\frac12}\xi_s(t)|_H \\ &\ge -e^{\Delta t_0}|\xi_s(t)|_H^2 - \frac{1}{2}|(-A)^{-\frac12}F(t)|_H^2,\end{aligned}$$

then by Gronwall's Lemma there exists $C(T,\Delta t_0) \in (0,\infty)$ such that for all $s \le t \le T$,

$$|\xi_s(t)|^2 \le C(T,\Delta t_0)\int_s^T |(-A)^{-1/2}F(r)|^2 dr.$$

Hence, with the inequalities $|(-A)^{-\frac12}\cdot|_H \le C|\cdot|_H$ and (4.16), we obtain (4.15), leading to estimate (4.14), concluding the proof of Lemma 4.4.

5 Proof of Theorem 3.2

The aim of this section is to prove Theorem 3.2. Let the numerical scheme $\big(X_n\big)_{n\in\mathbb{N}}$ be given by (2.12). The section is organized as follows. An auxiliary process $\tilde{X}$ and an appropriate decomposition of the error are given in Sect. 5.1. Error terms are estimated in Sect. 5.2. Auxiliary results are proved in Sect. 5.3.

5.1 Decomposition of the error

As explained in Sect. 3, the strategy for the weak error analysis requires to apply Itô's formula, hence the definition of an appropriate continuous-time process $\tilde{X}$.

Set, for every $n \in \mathbb{N}$, and every $t \in [n\Delta t, (n+1)\Delta t]$,

$$\tilde{X}(t) = e^{(t-n\Delta t)A}\Phi_{t-n\Delta t}(X_n) + \int_{n\Delta t}^{t} e^{(t-s)A} dW(s). \tag{5.1}$$

By construction, $\tilde{X}(n\Delta t) = X_n$ for all $n \in \mathbb{N}$. Moreover, for all $t \in [n\Delta t, (n+1)\Delta t]$, for $n \in \mathbb{N}$

$$d\tilde{X}(t) = A\tilde{X}(t)dt + dW(t) + e^{(t-n\Delta t)A}\Psi_0(\Phi_{t-n\Delta t}(X_n))dt.$$

Recall that $\Phi_{t-n\Delta t}$ and Ψ_0 are defined by (2.5).

The following result gives moment estimates. The proof is postponed to Sect. 5.3.

Lemma 5.1 *Let $T \in (0, \infty)$, $\Delta t_0 \in (0, 1]$ and $M \in \mathbb{N}$. There exists $C(T, \Delta t_0, M) \in (0, \infty)$ such that, for all $\Delta t \in (0, \Delta t_0]$ and $x \in E$,*

$$\sup_{t\in[0,T]} \mathbb{E}\big[|\tilde{X}(t)|_E^M\big] \le C(T, \Delta t_0, M)\big(1 + |x|_E^M\big).$$

The error is then decomposed as follows, using Itô's formula and the Kolmogorov equation (2.11), with $T = N\Delta t$

$$\begin{aligned}
&\mathbb{E}\big[u^{(\Delta t)}(T, x)\big] - \mathbb{E}\big[u^{(\Delta t)}(0, X_N)\big]\\
&= \sum_{k=0}^{N-1}\Big(\mathbb{E}\big[u^{(\Delta t)}((N-k)\Delta t, X_k)\big] - \mathbb{E}\big[u^{(\Delta t)}((N-k-1)\Delta t, X_{k+1})\big]\Big)\\
&= \sum_{k=0}^{N-1}\mathbb{E}\int_{k\Delta t}^{(k+1)\Delta t} \langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), \Psi_{\Delta t}(\tilde{X}(t))\\
&\quad - e^{(t-k\Delta t)A}\Psi_0(\Phi_{t-k\Delta t}(X_k))\rangle dt\\
&= \sum_{k=0}^{N-1}\big(d_k^1 + d_k^2 + d_k^3 + d_k^4\big),
\end{aligned}$$

where the error terms are defined by

$$
\begin{aligned}
d_k^1 &= \mathbb{E}\int_{k\Delta t}^{(k+1)\Delta t} \langle Du^{(\Delta t)}(T-t,\tilde{X}(t)), \Psi_{\Delta t}(\tilde{X}(t)) - \Psi_{\Delta t}(X_k)\rangle dt,\\
d_k^2 &= \mathbb{E}\int_{k\Delta t}^{(k+1)\Delta t} \langle Du^{(\Delta t)}(T-t,\tilde{X}(t)), \big(I - e^{(t-k\Delta t)A}\big)\Psi_{\Delta t}(X_k)\rangle dt,\\
d_k^3 &= \mathbb{E}\int_{k\Delta t}^{(k+1)\Delta t} \Big\langle Du^{(\Delta t)}(T-t,\tilde{X}(t)),\\
&\qquad e^{(t-k\Delta t)A}\big(\Psi_{\Delta t}(X_k) - \Psi_{\Delta t}(\Phi_{t-k\Delta t}(X_k))\big)\Big\rangle dt,\\
d_k^4 &= \mathbb{E}\int_{k\Delta t}^{(k+1)\Delta t} \Big\langle Du^{(\Delta t)}(T-t,\tilde{X}(t)),\\
&\qquad e^{(t-k\Delta t)A}\big(\Psi_{\Delta t}(\Phi_{t-k\Delta t}(X_k)) - \Psi_0(\Phi_{t-k\Delta t}(X_k))\big)\Big\rangle dt.
\end{aligned}
$$

5.2 Estimates of error terms

The objective of this section is to provide upper bounds for the error terms d_k^1, d_k^2, d_k^3 and d_k^4 defined above.

Most of the effort is devoted to the treatment of the error term d_k^1. Let $\eta > \frac{1}{4}$, $\alpha \in (0, \frac{1}{2})$, and $\varepsilon > 0$, such that $\alpha + 3\varepsilon < \frac{1}{2}$.

Assume $k = 0$, then, thanks to Theorem 4.1 and Lemmas 2.1 and 5.1

$$
|d_0^1| \le C(T,\Delta t_0)\|\varphi\|_{1,\infty}\Delta t(1+|x|_E^5).
$$

Assume that $1 \le k \le N-1$. Thanks to Theorem 4.1,

$$
\begin{aligned}
|d_k^1| &\le \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[|\langle Du^{(\Delta t)}(T-t,\tilde{X}(t)), \Psi_{\Delta t}(\tilde{X}(t)) - \Psi_{\Delta t}(X_k)\rangle|\big]dt\\
&\le \int_{k\Delta t}^{(k+1)\Delta t} \frac{C_{\eta+\frac{\alpha+\varepsilon}{2}}(T,\Delta t_0)}{(T-t)^{\eta+\frac{\alpha+\varepsilon}{2}}}\|\varphi\|_{1,\infty}\\
&\qquad \times \mathbb{E}\big[(1+|\tilde{X}(t)|_E^2)\big|(-A)^{-\eta-\frac{\alpha+\varepsilon}{2}}\big(\Psi_{\Delta t}(\tilde{X}(t)) - \Psi_{\Delta t}(X_k)\big)\big|_H\big]dt.
\end{aligned}
$$

Thanks to Lemma 2.2 and Taylor's formula,

$$
\big|(-A)^{-\eta-\frac{\alpha+\varepsilon}{2}}\big(\Psi_{\Delta t}(\tilde{X}(t)) - \Psi_{\Delta t}(X_k)\big)\big|_H \le \mathscr{M}(X_k,\tilde{X}(t))\big|(-A)^{-\frac{\alpha}{2}}(\tilde{X}(t) - X_k)\big|_H,
$$

with

$$
\begin{aligned}
&\mathscr{M}(X_k,\tilde{X}(t))\\
&\quad \le C_{\alpha,\varepsilon}(\Delta t_0)\Big(1 + |\tilde{X}(t)|_E^2 + |(-A)^{\frac{\alpha+2\varepsilon}{2}}\tilde{X}(t)|_H^2 + |X_k|_E^2 + |(-A)^{\frac{\alpha+2\varepsilon}{2}}X_k|_H^2\Big).
\end{aligned}
$$

An application of Hölder's inequality shows that it is required to prove upper bounds for $\big(\mathbb{E}\mathscr{M}(X_k, \tilde{X}(t))^4\big)^{\frac{1}{4}}$ and $\big(\mathbb{E}\big|(-A)^{-\frac{\alpha}{2}}(\tilde{X}(t) - X_k)\big|_H^2\big)^{\frac{1}{2}}$. The following auxiliary results give the required estimates. The proofs are postponed to Sect. 5.3.

Lemma 5.2 *Let $T \in (0,\infty)$, $\Delta t_0 \in (0,1]$, $\alpha \in [0, \frac{1}{2})$ and $M \in \mathbb{N}$. There exists a constant $C_\alpha(T, \Delta t_0, M) \in (0,\infty)$ such that, for all $\Delta t \in (0, \Delta t_0]$, $x \in E$, and $n \in \mathbb{N}$ with $n\Delta t \le T$,*

$$\begin{aligned}&\big(\mathbb{E}[|(-A)^{\frac{\alpha}{2}} W^A(n\Delta t)|_H^M]\big)^{\frac{1}{M}} + \big(\mathbb{E}[|(-A)^{\frac{\alpha}{2}} X_n|_H^M]\big)^{\frac{1}{M}}\\ &\quad\le C_\alpha(T, \Delta t_0, M)(1 + |x|_E + (n\Delta t)^{-\frac{\alpha}{2}}|x|_H).\end{aligned}$$

Lemma 5.3 *Let $T \in (0,\infty)$, $\Delta t_0 \in (0,1]$, $\alpha \in [0, \frac{1}{2})$. There exists $C_\alpha(T, \Delta t_0) \in (0,\infty)$ such that, for all $\Delta t \in (0, \Delta t_0]$ and $x \in E$, for all $t \in [0,T] \cap [n\Delta t, (n+1)\Delta t]$, $n \in \mathbb{N}$,*

$$\big(\mathbb{E}|(-A)^{-\frac{\alpha}{2}}(\tilde{X}(t) - X_n)|_H^2\big)^{\frac{1}{2}} \le C_\alpha(T, \Delta t_0)\Delta t^\alpha(1 + |x|_E^3 + (n\Delta t)^{-\frac{\alpha}{2}}|x|_H).$$

The result of Lemma 5.3 may be interpreted as follows: whereas the trajectories are only $\frac{\alpha}{2}$-Hölder continuous with values in H, they are α-Hölder continuous when measured in $|(-A)^{-\frac{\alpha}{2}} \cdot|$-norm, when $\alpha < \frac{1}{2}$. In the context of this section, this property explains why the weak order is twice the strong order.

Let us now apply the results of Lemmas 5.2 and 5.3 to treat the error term d_k^1. Thanks to Lemma 2.1, $\Phi_{\Delta t}$ is globally Lipschitz continuous, hence applying (2.3) yields

$$\begin{aligned}|(-A)^{\frac{\alpha+2\varepsilon}{2}}\tilde{X}(t)|_H &\le |(-A)^{\frac{\alpha+2\varepsilon}{2}}\Phi_{t-n\Delta t}(X_n)|_H\\ &\quad + |(-A)^{\frac{\alpha+2\varepsilon}{2}}\big(W^A(t) - e^{(t-n\Delta t)A}W^A(n\Delta t)\big)|_H\\ &\le Ce^{\Delta t}|(-A)^{\frac{\alpha+3\varepsilon}{2}}X_n|_H + |(-A)^{\frac{\alpha+2\varepsilon}{2}}W^A(t)|\\ &\quad + |(-A)^{\frac{\alpha+2\varepsilon}{2}}W^A(n\Delta t)|_H.\end{aligned}$$

With the condition $\alpha + 3\varepsilon < \frac{1}{2}$, Lemma 5.2 above then implies

$$\big(\mathbb{E}\mathscr{M}(X_k, \tilde{X}(t))^4\big)^{\frac{1}{4}} \le C(1 + |x|_E^2 + (k\Delta t)^{-\alpha-3\varepsilon}|x|_H^2).$$

Finally, using Lemmas 5.1 and 5.3, for $1 \le k \le n-1$,

$$|d_k^1| \le \Delta t^\alpha \int_{k\Delta t}^{(k+1)\Delta t} \frac{C_{\alpha,\varepsilon}(T, \Delta t_0)}{(T-t)^{\eta + \frac{\alpha+\varepsilon}{2}}}\|\varphi\|_{1,\infty} dt \left(1 + \frac{1}{(k\Delta t)^{2\alpha+3\varepsilon}}\right)(1 + |x|_E^5).$$

Observe that $2\alpha + 3\varepsilon < 1$. This concludes the treatment of the error term d_k^1.

It remains to treat the error terms d_k^2, d_k^3, d_k^4. Compared with the arguments used to treat d_k^1, the computations are simpler.

First, let $\alpha \in (0, \frac{1}{2})$. Then, thanks to Theorem 4.1, for all $k \in \{0, \ldots, N-1\}$, one has

$$\begin{aligned}
|d_k^2| &\le \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[|\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), \big(I - e^{(t-k\Delta t)A}\big)\Psi_{\Delta t}(X_k)\rangle|\big]dt \\
&\le \int_{k\Delta t}^{(k+1)\Delta t} \frac{C_{2\alpha}(T, \Delta t_0)\|\varphi\|_{1,\infty}}{(T-t)^{2\alpha}} \|(-A)^{-2\alpha}(I - e^{(t-k\Delta t)A})\|_{\mathcal{L}(H)} \\
&\quad \times \mathbb{E}\big[(1 + |\tilde{X}(t)|_E^2)|\Psi_{\Delta t}(X_k)|_H\big]dt \\
&\le \Delta t^{2\alpha} \int_{k\Delta t}^{(k+1)\Delta t} \frac{C_{2\alpha}(T, \Delta t_0)\|\varphi\|_{1,\infty}}{(T-t)^{2\alpha}}(1 + |x|_E^5)dt,
\end{aligned}$$

using the chain of inequalities $\|(-A)^{-2\alpha}(I - e^{(t-k\Delta t)A})\|_{\mathcal{L}(H)} \le C_{2\alpha}(t - k\Delta t)^{2\alpha} \le C_{2\alpha}\Delta t^{2\alpha}$ when $0 \le t - k\Delta t \le \Delta t$. This concludes the treatment of d_k^2.

Second, thanks to Lemma 2.1, for all $z \in \mathbb{R}$, and all $0 \le \tau \le \Delta t \le \Delta t_0$, one has

$$\begin{aligned}
|\Psi_{\Delta t}(z) - \Psi_{\Delta t}(\Phi_\tau(z))| &\le C(\Delta t_0)|z - \Phi_\tau(z)|(1 + |z|^2) \\
&\le C(\Delta t_0)\tau|\Psi_\tau(z)|(1 + |z|^2) \\
&\le C(\Delta t_0)\Delta t(1 + |z|^5).
\end{aligned}$$

Using Theorem 4.1, for all $k \in \{0, \ldots, N-1\}$, one obtains

$$\begin{aligned}
|d_k^3| &\le \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[|\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), \\
&\qquad e^{(t-k\Delta t)A}\big(\Psi_{\Delta t}(X_k) - \Psi_{\Delta t}(\Phi_{t-k\Delta t}(X_k))\big)\rangle|\big]dt \\
&\le \int_{k\Delta t}^{(k+1)\Delta t} C_0(T, \Delta t_0)\|\varphi\|_{1,\infty}\mathbb{E}\big[\big|\Psi_{\Delta t}(X_k) - \Psi_{\Delta t}(\Phi_{t-k\Delta t}(X_k))\big|_H\big]dt \\
&\le \Delta t \int_{k\Delta t}^{(k+1)\Delta t} C_0(T, \Delta t_0)\|\varphi\|_{1,\infty}\mathbb{E}\big[(1 + |X_k|_E^5)\big]dt \\
&\le \Delta t^2 C(T, \Delta t_0)\|\varphi\|_{1,\infty}(1 + |x|_E^5).
\end{aligned}$$

This concludes the treatment of d_k^3.

Finally, thanks to Theorem 4.1 and Lemma 2.1, for all $k \in \{0, \ldots, N-1\}$, one obtains

$$\begin{aligned}
|d_k^4| &\le \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[|\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), e^{(t-k\Delta t)A} \\
&\quad \times \big(\Psi_{\Delta t}(\Phi_{t-k\Delta t}(X_k)) - \Psi_0(\Phi_{t-k\Delta t}(X_k))\big)\rangle|\big]dt \\
&\le \int_{k\Delta t}^{(k+1)\Delta t} C_0(T, \Delta t_0)\|\varphi\|_{1,\infty}\mathbb{E}\big[\big|\Psi_{\Delta t}(\Phi_{t-k\Delta t}(X_k)) - \Psi_0(\Phi_{t-k\Delta t}(X_k))\big|_H\big]dt
\end{aligned}$$

$$\leq \Delta t \int_{k\Delta t}^{(k+1)\Delta t} C_0(T, \Delta t_0)\|\varphi\|_{1,\infty}\mathbb{E}\big[(1 + |X_k|_E^5)\big]dt$$
$$\leq \Delta t^2 C(T, \Delta t_0)\|\varphi\|_{1,\infty}\big(1 + |x|_E^5\big).$$

This concludes the treatment of the last error term d_k^4.

We now proceed to conclude the proof of Theorem 3.2. Using the decomposition of the weak error obtained in Sect. 5.1,

$$\mathbb{E}\big[u^{(\Delta t)}(T, x)\big] - \mathbb{E}\big[u^{(\Delta t)}(0, X_N)\big] = \sum_{k=0}^{N-1}\big(d_k^1 + d_k^2 + d_k^3 + d_k^4\big),$$

and gathering the estimates for the error terms, one has

$$\begin{aligned}
&\big|\mathbb{E}\big[u^{(\Delta t)}(T, x)\big] - \mathbb{E}\big[u^{(\Delta t)}(0, X_N)\big]\big| \leq C(T, \Delta t_0, |x|_E)\|\varphi\|_{1,\infty} \\
&\quad\times\Big(\Delta t + \Delta t^\alpha \sum_{k=1}^{N-1}\int_{k\Delta t}^{(k+1)\Delta t} \frac{C_{\alpha,\varepsilon}(T, \Delta t_0)}{(T-t)^{\eta+\frac{\alpha+\varepsilon}{2}} t^{2\alpha+3\varepsilon}} dt \\
&\quad+\Delta t^{2\alpha}\sum_{k=0}^{N-1}\int_{k\Delta t}^{(k+1)\Delta t}\frac{C_{2\alpha}(T, \Delta t_0)\|\varphi\|_{1,\infty}}{(T-t)^{2\alpha}}dt + \Delta t(N\Delta t)\Big) \\
&\leq C(T, \Delta t_0, |x|_E)\|\varphi\|_{1,\infty}\Delta t^\alpha,
\end{aligned}$$

where $\alpha < \frac{1}{2}$ is arbitrary close to $\frac{1}{2}$, and the auxiliary parameter $\varepsilon > 0$ is chosen such that $\alpha + 3\varepsilon < \frac{1}{2}$. Combined with the arguments of Sect. 3, this concludes the proof of Theorem 3.2.

5.3 Proof of Lemmas 5.1, 5.2 and 5.3

It remains to provide the proofs of the moment estimates (Lemma 5.1), and of the regularity properties (Lemmas 5.2 and 5.3) used to treat the error term d_k^1 in Sect. 5.2.

Proof of Lemma 5.1 For any $n \in \mathbb{N}$, and $t \in [n\Delta t, (n+1)\Delta t]$, the definition (5.1) of $\tilde{X}(t)$ gives

$$\begin{aligned}
|\tilde{X}(t)|_E &\leq |\Phi_{t-n\Delta t}(X_n)|_E + |W^A(t) - e^{(t-n\Delta t)A}W^A(n\Delta t)|_E \\
&\leq e^{\Delta t}|X_n|_E + |W^A(t)|_E + |W^A(n\Delta t)|_E,
\end{aligned}$$

thanks to Lemma 2.1. Using (2.9) and Lemma 2.4 then concludes the proof. □

Proof of Lemma 5.2 Note that, for all $n \in \mathbb{N}$, such that $n\Delta t \leq T$,

$$\begin{aligned}
&\big(\mathbb{E}|(-A)^{\frac{\alpha}{2}}X_n|_H^M\big)^{\frac{1}{M}} \\
&\leq |(-A)^{\frac{\alpha}{2}}e^{n\Delta t A}x|_H + C(T)\Delta t\sum_{k=0}^{n-1}\Big(\mathbb{E}|(-A)^{\frac{\alpha}{2}}e^{(n-k)\Delta t}\Psi_{\Delta t}(X_k)|_H^M\Big)^{\frac{1}{M}}
\end{aligned}$$

$$
\begin{aligned}
&+ \big(\mathbb{E}|(-A)^{\frac{\alpha}{2}} W^A(n\Delta t)|_H^M)\big)^{\frac{1}{M}} \\
&\leq (n\Delta t)^{-\frac{\alpha}{2}} |x|_E + C(T)\Delta t \sum_{k=0}^{n-1} \frac{1}{((n-k)\Delta t)^{\frac{\alpha}{2}}} \big(\mathbb{E}|\Psi_{\Delta t}(X_k)|_H^M\big)^{\frac{1}{M}} + C_\alpha(T, M),
\end{aligned}
$$

since, for all $t \in [0, T]$,

$$
\begin{aligned}
\mathbb{E}|(-A)^{\frac{\alpha}{2}} W^A(t)|_H^M &\leq C \left(\int_0^t \|(-A)^{\frac{\alpha}{2}} e^{(t-s)A}\|_{\mathscr{L}_2(H)}^2 ds \right)^{\frac{M}{2}} \\
&\leq C \left(\int_0^t (t-s)^{-\alpha-\frac{1}{2}-\varepsilon} ds \right)^{\frac{M}{2}} \\
&\leq C_\alpha(T, M),
\end{aligned}
$$

where $\varepsilon \in (0, \frac{1}{2} - \alpha)$. Thanks to Lemma 5.1, then

$$
\big(\mathbb{E}|(-A)^{\frac{\alpha}{2}} X_n|_H^M\big)^{\frac{1}{M}} \leq C_\alpha(T, \Delta t_0, M)(1 + |x|_E + (n\Delta t)^{-\frac{\alpha}{2}} |x|_H).
$$

This concludes the proof of Lemma 5.2. □

Proof of Lemma 5.3 For $t \in [n\Delta t, (n+1)\Delta t]$, $t \leq T$,

$$
\begin{aligned}
\tilde{X}(t) - X_n = e^{(t-n\Delta t)A} X_n - X_n + (t - n\Delta t) e^{(t-n\Delta t)A} \Psi_{t-n\Delta t}(X_n) \\
+ \int_{n\Delta t}^t e^{(t-s)A} dW(s).
\end{aligned}
$$

First,

$$
\begin{aligned}
&\big(\mathbb{E}|(-A)^{-\frac{\alpha}{2}} (e^{(t-n\Delta t)A} - I) X_n|_H^2\big)^{\frac{1}{2}} \\
&\quad \leq \|(-A)^{-\alpha}(e^{(t-n\Delta t)A} - I)\|_{\mathscr{L}(H)} \big(\mathbb{E}|(-A)^{\frac{\alpha}{2}} X_n|_H^2\big)^{\frac{1}{2}} \\
&\quad \leq C_\alpha(T, \Delta t_0) \Delta t^\alpha (1 + |x|_E + (n\Delta t)^{-\frac{\alpha}{2}} |x|_H),
\end{aligned}
$$

thanks to Lemma 5.2. Second,

$$
\begin{aligned}
\Big(\mathbb{E}\big|(-A)^{-\frac{\alpha}{2}} (t - n\Delta t) e^{(t-n\Delta t)A} \Psi_{t-n\Delta t}(X_n)\big|_H^2\Big)^{\frac{1}{2}} &\leq \Delta t \big(\mathbb{E}|\Psi_{t-n\Delta t}(X_n)|_E^2\big)^{\frac{1}{2}} \\
&\leq C(T, \Delta t_0) \Delta t (1 + |x|_E^3),
\end{aligned}
$$

thanks to Lemma 5.1. Third, by Itô's formula, for $\varepsilon = \frac{1}{2} - \alpha > 0$,

$$\begin{aligned}
\mathbb{E}|(-A)^{-\frac{\alpha}{2}} \int_{n\Delta t}^{t} e^{(t-s)A} dW(s)|_H^2 &= \int_{n\Delta t}^{t} \|(-A)^{-\frac{\alpha}{2}} e^{(t-s)A}\|_{\mathscr{L}_2(H)}^2 ds \\
&\leq C_\alpha \int_{n\Delta t}^{t} (t-s)^{-\frac{1}{2}-\varepsilon+\alpha} ds \\
&\leq C_\alpha \Delta t^{\frac{1}{2}-\varepsilon+\alpha} = C_\alpha \Delta t^{2\alpha}.
\end{aligned}$$

This concludes the proof of Lemma 5.3. □

6 Proof of Theorem 3.3

The aim of this section is to prove Theorem 3.3. Let the numerical scheme $\big(X_n\big)_{n\in\mathbb{N}}$ be given by (2.13).

The section is organized as follows. An auxiliary process $\tilde{X}$ and an appropriate decomposition of the error are given in Sect. 6.1. Error terms are estimated in Sect. 6.2. Auxiliary results are proved in Sect. 6.3.

Assume that φ satisfies Assumption 3.1. By linearity of the error with respect to φ, without loss of generality, it is assumed that $\|\varphi\|_{2,\infty} \leq 1$, in order to simplify the notation.

6.1 Decomposition of the error

As explained in Sect. 3, the strategy for the weak error analysis requires to apply Itô's formula, hence the definition of an appropriate continuous-time process $\tilde{X}$.

Set, for every $n \in \mathbb{N}$, and every $t \in [n\Delta t, (n+1)\Delta t]$,

$$\tilde{X}(t) = X_n + (t - n\Delta t) A S_{\Delta t} X_n + (t - n\Delta t) S_{\Delta t} \Psi_{\Delta t}(X_n) + S_{\Delta t}\big(W(t) - W(n\Delta t)\big), \tag{6.1}$$

where we recall that $S_{\Delta t} = (I - \Delta t A)^{-1}$.

By construction, $\tilde{X}(n\Delta t) = X_n$ for all $n \in \mathbb{N}$. Moreover,

$$d\tilde{X}(t) = A S_{\Delta t} X_n dt + S_{\Delta t} \Psi_{\Delta t}(X_n) dt + S_{\Delta t} dW(t), \quad t \in [n\Delta t, (n+1)\Delta t], n \in \mathbb{N}.$$

The following result gives moment estimates. The proof is postponed to Sect. 6.1.

Lemma 6.1 *Let $T \in (0,\infty)$, $\Delta t_0 \in (0,1]$ and $M \in \mathbb{N}$. There exists $C(T, \Delta t_0, M) \in (0,\infty)$ such that, for all $\Delta t \in (0, \Delta t_0]$ and $x \in E$,*

$$\sup_{t\in[0,T]} \mathbb{E}[|\tilde{X}(t)|_E^M] \leq C(T, \Delta t_0, M)(1 + |x|_E^3)^M.$$

The error is then decomposed as follows, using Itô's formula and the Kolmogorov equation (2.11), with $T = N\Delta t$,

$$
\begin{aligned}
&\mathbb{E}[u^{(\Delta t)}(T,x)] - \mathbb{E}[u^{(\Delta t)}(0,X_N)] \\
&\quad = \mathbb{E}[u^{(\Delta t)}(T,x)] - \mathbb{E}[u^{(\Delta t)}(T-\Delta t, X_1)] \\
&\qquad + \sum_{k=1}^{N-1} \Big(\mathbb{E}\big[u^{(\Delta t)}((N-k)\Delta t, X_k)\big] - \mathbb{E}\big[u^{(\Delta t)}((N-k-1)\Delta t, X_{k+1})\big]\Big) \\
&\quad = \mathbb{E}[u^{(\Delta t)}(T-\Delta t, X^{(\Delta t)}(\Delta t))] - \mathbb{E}[u^{(\Delta t)}(T-\Delta t, X_1)] \\
&\qquad + \sum_{k=1}^{N-1} \big(a_k + b_k + c_k\big),
\end{aligned}
$$

where

$$
\begin{aligned}
a_k &= \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), A\tilde{X}(t) - AS_{\Delta t}X_k\rangle dt \\
b_k &= \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), \Psi_{\Delta t}(\tilde{X}(t)) - S_{\Delta t}\Psi_{\Delta t}(X_k)\rangle dt \\
c_k &= \frac{1}{2}\int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\left[\sum_{j\in\mathbb{N}} D^2u^{(\Delta t)}(T-t, \tilde{X}(t)).(e_j, e_j)\left(1 - \frac{1}{(1+\lambda_j\Delta t)^2}\right)\right] dt.
\end{aligned}
$$

Section 6.2 is devoted to the proof of Lemmas 6.2 and 6.3 below. Theorem 3.3 is a straightforward consequence of these results, thanks to the decomposition of the error above.

Lemma 6.2 *Let $T \in (0,\infty)$, $\Delta t_0 \in (0,1]$ and $x \in E$. For all $\alpha \in [0, \frac{1}{2})$, there exists $C_\alpha(T, \Delta t_0, |x|_E) \in (0,\infty)$ such that, for all $\Delta t \in (0, \Delta t_0]$,*

$$
\left|\mathbb{E}[u^{(\Delta t)}(T-\Delta t, X^{(\Delta t)}(\Delta t))] - \mathbb{E}[u^{(\Delta t)}(T-\Delta t, X_1)]\right| \le C_\alpha(T, \Delta t_0, |x|_E)\Delta t^\alpha.
$$

Lemma 6.3 *Let $T \in (0,\infty)$, $\Delta t_0 \in (0,1]$ and $x \in E$. For all $\alpha \in [0, \frac{1}{2})$, there exists $C_\alpha(T, \Delta t_0, |x|_E) \in (0,\infty)$ such that, for all $\Delta t \in (0, \Delta t_0]$,*

$$
\sum_{k=1}^{n-1}(|a_k| + |b_k| + |c_k|) \le C_\alpha(T, \Delta t_0, |x|_E)\Delta t^\alpha.
$$

Compared with Sect. 5, there are more error tems, and the analysis is more technical. The proof essentially follows the same strategy as in [13]. In particular, Malliavin calculus techniques [27] are employed. For completeness, details are given.

We emphasize that the most important new result is the estimate on the Malliavin derivative, see Lemma 6.4. This result is non trivial, since $\Psi_{\Delta t}$ is not globally Lipschitz continuous, and it is obtained thanks to the structure of the numerical scheme, based on a splitting approach.

Note that an alternative approach to treat the term b_k below, would be to adapt the strategy used in Sect. 5.2 to treat the term d_k^1, using Lemma 2.2 in particular.

Appropriate versions of Lemmas 5.2 and 5.3 would be required. It seems that this alternative approach does not considerably shorten the proof.

We recall the Malliavin calculus duality formula in some Hilbert space K, see [27]. Let $\mathbb{D}^{1,2}$ be the closure of smooth random variables (with respect to Malliavin derivative) for the topology defined by the norm

$$\|F\|_{\mathbb{D}^{1,2}} = \left(\mathbb{E}[|F|_K^2] + \mathbb{E}\left[\int_0^T |\mathscr{D}_s F|_K^2 ds\right]\right)^{\frac{1}{2}}.$$

where $\mathscr{D}_s F$ denotes the Malliavin derivative of F. For $F \in \mathbb{D}^{1,2}$ and $\Xi \in L^2(\Omega \times [0,T]; K)$ such that $\Xi(t) \in \mathbb{D}^{1,2}$ for all $t \in [0,T]$ and $\int_0^T \int_0^T |\mathscr{D}_s \Xi(t)|^2 ds dt < +\infty$, we have the integration by parts formula:

$$\mathbb{E}\left[F \int_0^T \langle \Xi(s), dW(s)\rangle\right] = \mathbb{E}\left[\int_0^T \langle \mathscr{D}_s F, \Xi(s)\rangle ds\right].$$

In our context, we will use another form of this integration by parts formula, see [13, Lemma 2.1]: for $u \in \mathscr{C}_b^2(H)$, and any adapted process $\Xi \in L^2(\Omega \times [0,T]; \mathscr{L}_2(H))$,

$$\mathbb{E}\left[\langle Du^{(\Delta t)}(F), \int_0^T \Xi(s) dW(s)\rangle\right] = \mathbb{E}\left[\sum_{j\in\mathbb{N}} \int_0^T D^2 u^{(\Delta t)}(F).(\mathscr{D}_s^{e_j} F, \Xi(s)e_j) ds\right]. \tag{6.2}$$

6.2 Estimates of error terms

This section is decomposed into five parts. First, the error term appearing in Lemma 6.2 needs to be treated separately from the others. Then, the error terms a_k, b_k and c_k are treated successively. Finally, the conclusion is given in the last part by gathering all the error estimates. The structure of the proof is very similar to the one in [13], with some extra care devoted to the treatment of the non globally Lipschitz nonlinearity.

6.2.1 Proof of Lemma 6.2

The fundamental idea is to apply the regularization property for the first order derivative $Du^{(\Delta t)}$. This leads to consider the strong error $\mathbb{E}|(-A)^{-2\alpha}(X^{(\Delta t)}(\Delta t) - X_1)|_H$ in a weaker norm, hence a better rate of convergence.

For any $\alpha \in [0, \frac{1}{2})$, thanks to Theorem 4.1, one has

$$\begin{aligned}
&\left|\mathbb{E}[u^{(\Delta t)}(T - \Delta t, X^{(\Delta t)}(\Delta t))] - \mathbb{E}[u^{(\Delta t)}(T - \Delta t, X_1)]\right| \\
&\quad \leq \frac{C_\alpha}{(T - \Delta t)^{2\alpha}} \mathbb{E}|(-A)^{-2\alpha}(X^{(\Delta t)}(\Delta t) - X_1)|_H.
\end{aligned}$$

The right-hand side is decomposed into three contributions as follows:

$$
\begin{aligned}
\mathbb{E}|(-A)^{-2\alpha}(X^{(\Delta t)}(\Delta t)-X_1)|_H &\le |(-A)^{-2\alpha}\big(e^{\Delta t A}-S_{\Delta t}\big)x|_H \\
&+\int_0^{\Delta t}\mathbb{E}|(-A)^{-2\alpha}e^{(\Delta t-t)A}\Psi_{\Delta t}(X^{(\Delta t)}(t))|_H dt+\Delta t|(-A)^{-2\alpha}S_{\Delta t}\Psi_{\Delta t}(x)|_H \\
&+\mathbb{E}|\int_0^{\Delta t}(-A)^{-2\alpha}e^{(\Delta t-t)A}dW(t)|_H+\mathbb{E}|\int_0^{\Delta t}(-A)^{-2\alpha}S_{\Delta t}dW(t)|_H.
\end{aligned}
$$

First, one has

$$
\begin{aligned}
&|(-A)^{-2\alpha}\big(e^{\Delta t A}-S_{\Delta t}\big)x|_H \\
&\le \big(\|(-A)^{-2\alpha}(e^{\Delta t A}-I)\|_{\mathscr{L}(H)}+\|(-A)^{-2\alpha}(S_{\Delta t}-I)\|_{\mathscr{L}(H)}\big)|x|_H \\
&\le C_\alpha \Delta t^{2\alpha}|x|_H.
\end{aligned}
$$

Second, observe that $\|(-A)^{-2\alpha}\|_{\mathscr{L}(H)}<\infty$ for $\alpha\ge 0$, then using Lemma 2.3 gives

$$
\mathbb{E}|(-A)^{-\alpha}e^{(\Delta t-t)A}\Psi_{\Delta t}(X^{(\Delta t)}(t))|_H\le \mathbb{E}|\Psi_{\Delta t}(X^{(\Delta t)}(t))|_H\le C(1+|x|_E^3)
$$

and

$$
|(-A)^{-\alpha}S_{\Delta t}\Psi_{\Delta t}(x)|_H\le |\Psi_{\Delta t}(x)|_H\le C(1+|x|_E^3).
$$

Finally, for the stochastic integral terms, one obtains

$$
\begin{aligned}
&\mathbb{E}|\int_0^{\Delta t}(-A)^{-2\alpha}e^{(\Delta t-t)A}dW(t)|_H^2+\mathbb{E}|\int_0^{\Delta t}(-A)^{-2\alpha}S_{\Delta t}dW(t)|_H^2 \\
&\le 2\Delta t\|(-A)^{-2\alpha}\|_{\mathscr{L}_2(H)}^2
\end{aligned}
$$

owing to the property $\|(-A)^{-2\alpha}\|_{\mathscr{L}_2(H)}^2<\infty$ when $2\alpha>\frac{1}{4}$.

Gathering the estimates for the three contributions exhibited above, it is then straightforward to conclude that, for $\alpha\in[0,\frac{1}{2})$,

$$
\big|\mathbb{E}[u^{(\Delta t)}(T-\Delta t, X^{(\Delta t)}(\Delta t))]-\mathbb{E}[u^{(\Delta t)}(T-\Delta t, X_1)]\big|\le C_\alpha(T,|x|_E)\Delta t^\alpha.
$$

This concludes the proof of Lemma 6.2.

6.2.2 Proof of Lemma 6.3, Part 1

The aim of this section is to prove, for $\alpha\in[0,\frac{1}{2})$, that

$$
\sum_{k=1}^{N-1}|a_k|\le C_\alpha(T,\Delta t_0,|x|_E)\Delta t^\alpha.
$$

First, an appropriate decomposition of a_k is introduced.

For that purpose, decompose a_k as follows:

$$a_k = a_k^1 + a_k^2,$$

where

$$a_k^1 = \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), A(I - S_{\Delta t})X_k\rangle\big]dt,$$
$$a_k^2 = \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), A(\tilde{X}(t) - X_k)\rangle\big]dt.$$

Using the formulation

$$X_k = S_{\Delta t}^k x + \Delta t \sum_{\ell=0}^{k-1} S_{\Delta t}^{k-\ell} \Psi_{\Delta t}(X_\ell) + \sum_{\ell=0}^{k-1} \int_{\ell\Delta t}^{(\ell+1)\Delta t} S_{\Delta t}^{k-\ell} dW(t),$$

and the identity $I - S_{\Delta t} = -\Delta t S_{\Delta t} A$, the expression a_k^1 is decomposed as

$$a_k^1 = a_k^{1,1} + a_k^{1,2} + a_k^{1,3},$$

where

$$a_k^{1,1} = -\Delta t \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), A^2 S_{\Delta t}^{k+1} x\rangle\big]dt,$$
$$a_k^{1,2} = -\Delta t \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), \Delta t \sum_{\ell=0}^{k-1} A^2 S_{\Delta t}^{k-\ell+1} \Psi_{\Delta t}(X_\ell)\rangle\big]dt,$$
$$a_k^{1,3} = -\Delta t \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), \sum_{\ell=0}^{k-1} \int_{\ell\Delta t}^{(\ell+1)\Delta t} A^2 S_{\Delta t}^{k-\ell+1} dW(t)\rangle\big]dt.$$

Using (6.1), the expression a_k^2 is decomposed as

$$a_k^2 = a_k^{2,1} + a_k^{2,2} + a_k^{2,3},$$

where

$$a_k^{2,1} = \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), (t - t_k) A^2 S_{\Delta t} X_k\rangle\big]dt,$$
$$a_k^{2,2} = \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), (t - t_k) A S_{\Delta t} \Psi_{\Delta t}(X_k)\rangle\big]dt,$$
$$a_k^{2,3} = \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), \int_{k\Delta t}^{t} A S_{\Delta t} dW(s)\rangle\big]dt.$$

We now proceed with the analysis of each of those error terms. Special attention is given to the error terms $a_k^{1,3}$ and $a_k^{2,3}$ which employ Malliavin calculus techniques, whereas the arguments for the other terms are more straightforward.

Treatment of $a_k^{1,1}$

Let $\alpha \in [0, \frac{1}{2})$ and $\varepsilon \in (0, 1-2\alpha)$. Thanks to Theorem 4.1 and Lemma 6.1,

$$\begin{aligned}
|a_k^{1,1}| &\leq C_{2\alpha+\varepsilon}(T, \Delta t_0, |x|_E)\Delta t \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\varepsilon}} dt |(-A)^{2-2\alpha-\varepsilon} S_{\Delta t}^{k+1} x|_H \\
&\leq C_{\alpha,\varepsilon}(T, \Delta t_0, |x|_E)\Delta t \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\varepsilon}} dt \\
&\quad \times \|(-A)^{1-\varepsilon} S_{\Delta t}^k\|_{\mathscr{L}(H)} \|(-A)^{1-2\alpha} S_{\Delta t}\|_{\mathscr{L}(H)} \\
&\leq C_{\alpha,\varepsilon}(T, \Delta t_0, |x|_E)\Delta t^{2\alpha} \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(k\Delta t)^{1-\varepsilon}(T-t)^{2\alpha+\varepsilon}} dt,
\end{aligned}$$

where we have used the standard inequalities $\|(-A)^{1-\varepsilon} S_{\Delta t}^k\|_{\mathscr{L}(H)} \leq C_\varepsilon (k\Delta t)^{-1+\varepsilon}$ and $\|(-A)^{1-2\alpha} S_{\Delta t}\|_{\mathscr{L}(H)} \leq C_\alpha \Delta t^{2\alpha-1}$.

Treatment of $a_k^{1,2}$

Let $\alpha \in [0, \frac{1}{2})$ and $\varepsilon \in (0, 1-2\alpha)$. Thanks to Theorem 4.1, Lemma 6.1, and Cauchy-Schwarz inequality,

$$\begin{aligned}
|a_k^{1,2}| \leq C_{2\alpha+\varepsilon}(T, \Delta t_0, |x|_E)\Delta t \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\varepsilon}} dt \Delta t \\
\times \sum_{\ell=0}^{k-1} \big(\mathbb{E}\big|(-A)^{2-2\alpha-\varepsilon} S_{\Delta t}^{k-\ell+1} \Psi_{\Delta t}(X_\ell)\big|_H^2\big)^{\frac{1}{2}} dt.
\end{aligned}$$

Thanks to Lemma 6.1,

$$\begin{aligned}
&\Delta t \sum_{\ell=0}^{k-1} \big(\mathbb{E}\big|(-A)^{2-2\alpha-\varepsilon} S_{\Delta t}^{k-\ell+1} \Psi_{\Delta t}(X_\ell)\big|_H^2\big)^{\frac{1}{2}} \\
&\leq \Delta t \sum_{\ell=0}^{k-1} \|(-A)^{1-\varepsilon} S_{\Delta t}^{k-\ell}\|_{\mathscr{L}(H)} \|(-A)^{1-2\alpha} S_{\Delta t}\|_{\mathscr{L}(H)} \big(\mathbb{E}|\Psi_{\Delta t}(X_\ell)|_H^2\big)^{\frac{1}{2}} \\
&\leq C_{\alpha,\varepsilon}(T, \Delta t_0, |x|_E)\Delta t^{2\alpha-1},
\end{aligned}$$

using $\Delta t \sum_{\ell=0}^{k-1} \frac{1}{((k-\ell)\Delta t)^{1-\varepsilon}} \leq C_\varepsilon < \infty$. Thus

$$|a_k^{1,2}| \leq C_{\alpha,\varepsilon}(T, \Delta t_0, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\varepsilon}} dt \ \Delta t^{2\alpha}.$$

Treatment of $a_k^{1,3}$

The Malliavin calculus duality formula (6.2) is applied, for fixed t, with $u = u^{(\Delta t)}(T - t, \cdot)$, $F = \tilde{X}(t)$, and $\Xi(s) = A^2 S_{\Delta t}^{k-\ell+1}$ for $\ell \Delta t \le s \le (\ell+1)\Delta t$. This yields the following alternative expression for $a_k^{1,3}$:

$$
\begin{aligned}
a_k^{1,3} &= -\Delta t \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), \sum_{\ell=0}^{k-1} \int_{\ell \Delta t}^{(\ell+1)\Delta t} A^2 S_{\Delta t}^{k-\ell+1} dW(s)\rangle\big] dt \\
&= -\Delta t \int_{k\Delta t}^{(k+1)\Delta t} \sum_{\ell=0}^{k-1} \int_{\ell\Delta t}^{(\ell+1)\Delta t} \sum_{j\in\mathbb{N}} \mathbb{E}\big[D^2 u^{(\Delta t)}(T-t, \tilde{X}(t)). \\
&\qquad (\mathscr{D}_s^{e_j} \tilde{X}(t), A^2 S_{\Delta t}^{k-\ell+1} e_j)\big] ds dt.
\end{aligned}
$$

Lemma 6.4 below provides the required estimate. Its proof is postponed to Sect. 6.3.

Lemma 6.4 *Let $T \in (0, \infty)$.*

For all $k \in \mathbb{N}$, such that $k\Delta t \le T$, and all $s \in [0, T]$, almost surely,

$$\|\mathscr{D}_s X_k\|_{\mathscr{L}(H)} \le e^T.$$

In addition, $\mathscr{D}_s X_k = 0$ if $k\Delta t \le s$.

Moreover, for all $0 \le s < k\Delta t \le t \le (k+1)\Delta t \le T$,

$$\|\mathscr{D}_s \tilde{X}(t)\|_{\mathscr{L}(H)} \le (3 + \Delta t |\Psi'_{\Delta t}(X_k)|_E) e^T.$$

Let $\alpha \in [0, \frac{1}{2})$, and let $\kappa \in (\frac{1}{2} - \alpha, 1 - 2\alpha)$ and $\varepsilon \in (0, 1 - 2\alpha - \kappa)$ be two auxiliary parameters. Then $2\alpha + \kappa + \varepsilon < 1$, and $\alpha + \kappa > \frac{1}{2}$. Thanks to Theorem 4.2, Lemma 6.4, and the moment estimates, by Lemma 6.1,

$$
\begin{aligned}
|a_k^{1,3}| &\le C_{0,2\alpha+\varepsilon+\kappa}(T, \Delta t_0, |x|_E) \Delta t \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\varepsilon+\kappa}} dt \\
&\qquad \times \sum_{\ell=0}^{k-1} \Delta t \sum_{j\in\mathbb{N}} |(-A)^{2-2\alpha-\varepsilon-\kappa} S_{\Delta t}^{k-\ell+1} e_j|_H \\
&\le C_{\alpha,\varepsilon,\kappa}(T, \Delta t_0, |x|_E) \Delta t \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\varepsilon+\kappa}} dt \sum_{j\in\mathbb{N}} \frac{\lambda_j^{1-2\alpha-\kappa}}{(1+\lambda_j \Delta t)} \\
&\le C_{\alpha,\varepsilon,\kappa}(T, \Delta t_0, |x|_E) \Delta t \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\varepsilon+\kappa}} dt \sum_{j\in\mathbb{N}} \frac{(\Delta t \lambda_j)^{1-\alpha}}{(1+\lambda_j \Delta t)} \frac{\Delta t^{\alpha-1}}{\lambda_j^{\alpha+\kappa}} \\
&\le C_{\alpha,\varepsilon}(T, \Delta t_0, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\varepsilon+\kappa}} dt \; \Delta t^{\alpha},
\end{aligned}
$$

using $\Delta t \sum_{\ell=0}^{k-1} \frac{1}{\left((k-\ell)\Delta t\right)^{1-\varepsilon}} \le C_\varepsilon < \infty$, and $\sum_{j\in\mathbb{N}} \frac{1}{\lambda_j^{\alpha+\kappa}} < \infty$.

Now that all the terms $a_k^{1,1}$, $a_k^{1,2}$ and $a_k^{1,3}$ have been treated, it remains to deal with the terms $a_k^{2,1}$, $a_k^{2,2}$ and $a_k^{2,3}$.

Treatment of $a_k^{2,1}$

Note that $(t-t_k)AS_{\Delta t} = \frac{(t-t_k)}{\Delta t}(S_{\Delta t} - I)$. As a consequence, it is sufficient to repeat the treatment of a_k^1 above, and to use $t - t_k \leq \Delta t$, to get the required estimate for $a_k^{2,1}$:

$$|a_k^{2,1}| \leq C_{\alpha,\varepsilon}(T, \Delta t_0, |x|_E)\Delta t^\alpha \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(k\Delta t)^{1-\varepsilon}(T-t)^{2\alpha+\varepsilon+\kappa}} dt.$$

Treatment of $a_k^{2,2}$

Let $\alpha \in [0, \frac{1}{2})$. Thanks to Theorem 4.1 and Lemma 6.1,

$$\begin{aligned}
|a_k^{2,2}| &\leq C_{2\alpha}(T, \Delta t_0, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \frac{|t-t_k|}{(T-t)^{2\alpha}} dt \big(\mathbb{E}|(-A)^{1-2\alpha} S_{\Delta t}\Psi_{\Delta t}(X_k)|_H^2\big)^{\frac{1}{2}} \\
&\leq C_{2\alpha}(T, \Delta t_0, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha}} dt \ \Delta t^{2\alpha}.
\end{aligned}$$

Treatment of $a_k^{2,3}$

Using the Malliavin calculus duality formula (6.2),

$$\begin{aligned}
a_k^{2,3} &= \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), \int_{k\Delta t}^{t} AS_{\Delta t} dW(s)\rangle\big] dt \\
&= \int_{k\Delta t}^{(k+1)\Delta t} \int_{k\Delta t}^{t} \sum_{j\in\mathbb{N}} \mathbb{E}\big[D^2 u^{(\Delta t)}(T-t, \tilde{X}(t)).\big(\mathscr{D}_s^{e_j}\tilde{X}(t), AS_{\Delta t}e_j\big)\big] ds dt.
\end{aligned}$$

Observe that $\mathscr{D}_s\tilde{X}(t) = S_{\Delta t}$ for $k\Delta t \leq s \leq t \leq (k+1)\Delta t$. Let $\alpha \in [0, \frac{1}{2})$, and let $\kappa \in (\frac{1}{2} - \alpha, 1 - 2\alpha)$ be an auxiliary parameter. Then $2\alpha + \kappa < 1$ and $\alpha + \kappa > \frac{1}{2}$. Thanks to Theorem 4.2 and Lemma 6.1,

$$\begin{aligned}
|a_k^{2,3}| &\leq \int_{k\Delta t}^{(k+1)\Delta t} \int_{k\Delta t}^{t} \mathbb{E} \sum_{j=1}^{\infty} \frac{\lambda_j}{(1+\lambda_j \Delta t)^2} |D^2 u^{(\Delta t)}(T-t, \tilde{X}(t)).(e_j, e_j)| ds dt \\
&\leq C_{0,2\alpha+\kappa}(T, \Delta t_0, |x|_E) \sum_{j=1}^{\infty} \frac{\Delta t \lambda_j}{(1+\lambda_j\Delta t)^2 \lambda_j^{2\alpha+\kappa}} \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\kappa}} dt
\end{aligned}$$

$$\leq C_{\alpha,\varepsilon}(T, \Delta t_0, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\kappa}} dt \sum_{j\in\mathbb{N}} \frac{(\lambda_j \Delta t)^{1-\alpha}}{(1+\lambda_j \Delta t)^2} \frac{\Delta t^\alpha}{\lambda_j^{\alpha+\kappa}}$$

$$\leq C_{\alpha,\kappa}(T, \Delta t_0, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^{2\alpha+\kappa}} dt \Delta t^\alpha,$$

using $\sum_{j\in\mathbb{N}} \frac{1}{\lambda_j^{\alpha+\kappa}} < \infty$.

Conclusion

Gathering estimates above, for all $\alpha \in [0, \frac{1}{2})$,

$$\sum_{k=1}^{N-1} |a_k| \leq C_\alpha(T, \Delta t_0, |x|_E) \Delta t^\alpha \sum_{k=1}^{N-1} \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{t^{\beta_1(\alpha)}(T-t)^{\beta_2(\alpha)}} dt,$$

with two parameters $\beta_1(\alpha), \beta_2(\alpha) \in [0, 1)$. Therefore

$$\sum_{k=1}^{N-1} \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{t^{\beta_1(\alpha)}(T-t)^{\beta_2(\alpha)}} dt \leq \int_0^T \frac{1}{t^{\beta_1(\alpha)}(T-t)^{\beta_2(\alpha)}} dt < \infty.$$

This concludes the first part of the proof of Lemma 6.3.

6.2.3 Proof of Lemma 6.3, Part 2

We now proceed with the analysis of the error term b_k.

The aim of this section is to prove, for $\alpha \in [0, \frac{1}{2})$, that

$$\sum_{k=1}^{N-1} |b_k| \leq C_\alpha(T, \Delta t_0, |x|_E) \Delta t^\alpha.$$

For that purpose, decompose b_k as follows:

$$b_k = b_k^1 + b_k^2,$$

with

$$b_k^1 = \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), (I - S_{\Delta t})\Psi_{\Delta t}(X_k)\rangle\big] dt,$$

$$b_k^2 = \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), \Psi_{\Delta t}(\tilde{X}(t)) - \Psi_{\Delta t}(X_k)\rangle\big] dt.$$

Introduce real-valued functions $\Psi_{\Delta t}^j(\cdot) = \langle \Psi_{\Delta t}(\cdot), e_j\rangle$, for all $j \in \mathbb{N}$. Using Itô's formula, for $t \in [k\Delta t, (k+1)\Delta t]$,

$$\begin{aligned}
&\Psi^j_{\Delta t}(\tilde{X}(t)) - \Psi^j_{\Delta t}(X_k)\\
&= \int_{k\Delta t}^t \frac{1}{2}\sum_{i\in\mathbb{N}} D^2\Psi^j_{\Delta t}(S_{\Delta t}e_i, S_{\Delta t}e_i)ds\\
&\quad + \int_{k\Delta t}^t \langle D\Psi^j_{\Delta t}(\tilde{X}(s)), S_{\Delta t}AX_k\rangle ds + \int_{k\Delta t}^t \langle D\Psi^j_{\Delta t}(\tilde{X}(s)), S_{\Delta t}\Psi_{\Delta t}(X_k)\rangle ds\\
&\quad + \int_{k\Delta t}^t \langle D\Psi^j_{\Delta t}(\tilde{X}(s)), S_{\Delta t}dW(s)\rangle.
\end{aligned}$$

This expansion gives the decomposition

$$b_k^2 = b_k^{2,1} + b_k^{2,2} + b_k^{2,3} + b_k^{2,4},$$

with

$$\begin{aligned}
b_k^{2,1} &= \int_{k\Delta t}^{(k+1)\Delta t}\int_{k\Delta t}^t \sum_{i\in\mathbb{N}} \frac{\mathbb{E}\big[\langle Du^{(\Delta t)}(T-t,\tilde{X}(t)), D^2\Psi_{\Delta t}(\tilde{X}(s)).(e_i,e_i)\rangle\big]}{(1+\lambda_i\Delta t)^2}dsdt,\\
b_k^{2,2} &= \int_{k\Delta t}^{(k+1)\Delta t}\int_{k\Delta t}^t \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t,\tilde{X}(t)), D\Psi_{\Delta t}(\tilde{X}(s)).(S_{\Delta t}AX_k)\rangle\big]dsdt,\\
b_k^{2,3} &= \int_{k\Delta t}^{(k+1)\Delta t}\int_{k\Delta t}^t \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t,\tilde{X}(t)), D\Psi_{\Delta t}(\tilde{X}(s)).(S_{\Delta t}\Psi_{\Delta t}(X_k))\rangle\big]dsdt,\\
b_k^{2,4} &= \int_{k\Delta t}^{(k+1)\Delta t}\mathbb{E}\Big[\sum_{j=1}^{\infty}\langle Du^{(\Delta t)}(T-t,\tilde{X}(t)), e_j\rangle \int_{k\Delta t}^t \langle D\Psi^j_{\Delta t}(\tilde{X}(s)), S_{\Delta t}dW(s)\rangle\Big]dt.
\end{aligned}$$

We now proceed with the analysis of the error terms b_k^1, $b_k^{2,1}$, $b_k^{2,2}$, $b_k^{2,3}$ and $b_k^{2,4}$. The most technical part consists in the treatment of $b_k^{2,2}$, it requires to introduce a further decomposition.

Treatment of b_k^1

Thanks to Theorem 4.1 and Lemma 6.1,

$$\begin{aligned}
|b_k^1| &\le C_{2\alpha}(T,\Delta t_0, x|_E)\int_{k\Delta t}^{(k+1)\Delta t}\frac{1}{(T-t)^{2\alpha}}dt\big(\mathbb{E}|(-A)^{-2\alpha}(I-S_{\Delta t})\Psi_{\Delta t}(X_k)|_H^2\big)^{\frac{1}{2}}\\
&\le C_{2\alpha}(T,\Delta t_0,|x|_E)\int_{k\Delta t}^{(k+1)\Delta t}\frac{1}{(T-t)^{2\alpha}}dt\ \Delta t^{2\alpha}.
\end{aligned}$$

Treatment of $b_k^{2,1}$

Let $\alpha\in[0,\frac{1}{2})$. Thanks to Theorem 4.1 and Lemma 6.1, and to the inequality

$$|D^2\Psi_{\Delta t}(x).(e_i,e_i)|_H \le C(1+|x|_E)^M|e_i|_E^2,$$

we obtain

$$|b_k^{2,1}| \leq C_0(T, \Delta t_0, |x|_E) \sum_{i\in\mathbb{N}} \frac{\Delta t^2}{(1+\Delta t\lambda_i)^2} \leq C_\alpha(T, \Delta t_0, |x|_E) \sum_{i\in\mathbb{N}} \frac{1}{\lambda_i^{1-\alpha}} \Delta t^{1+\alpha}.$$

Treatment of $b_k^{2,2}$

A decomposition $b_k^{2,2} = b_k^{2,2,1} + b_k^{2,2,2} + b_k^{2,2,3}$ into three terms is required:

$$\begin{aligned}
b_k^{2,2,1} &= \int_{k\Delta t}^{(k+1)\Delta t} \int_{k\Delta t}^{t} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), D\Psi_{\Delta t}(\tilde{X}(s)).(AS_{\Delta t}^{k+1}x)\rangle\big] ds dt,\\
b_k^{2,2,2} &= \int_{k\Delta t}^{(k+1)\Delta t} \int_{k\Delta t}^{t} \mathbb{E}\Big[\Big\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)),\\
&\qquad D\Psi_{\Delta t}(\tilde{X}(s)).\Big(\Delta t \sum_{\ell=0}^{k-1} AS_{\Delta t}^{k-\ell+1}\Psi_{\Delta t}(X_\ell)\Big)\Big\rangle\Big] ds dt,\\
b_k^{2,2,3} &= \int_{k\Delta t}^{(k+1)\Delta t} \int_{k\Delta t}^{t} \mathbb{E}\Big[\Big\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)),\\
&\qquad D\Psi_{\Delta t}(\tilde{X}(s)).\Big(\sum_{\ell=0}^{k-1} \int_{\ell\Delta t}^{(\ell+1)\Delta t} AS_{\Delta t}^{k-\ell+1} dW(r)\Big)\Big\rangle\Big] ds dt.
\end{aligned}$$

Let $\alpha \in [0, \frac{1}{2})$. Thanks to Theorem 4.1, Lemma 6.1, and to inequalities already used above, the terms $b_k^{2,2,1}$ and $b_k^{2,2,2}$ are treated as follows. First,

$$\begin{aligned}
|b_k^{2,2,1}| &\leq C_0(T, \Delta t_0, |x|_E)\Delta t^2 \|A^{2\alpha} S_{\Delta t}^k\|_{\mathscr{L}(H)} \|A^{1-2\alpha} S_{\Delta t}\|_{\mathscr{L}(H)} |x|_H\\
&\leq C_\alpha(T, \Delta t_0, |x|_E)\Delta t^{1+2\alpha} \frac{1}{(k\Delta t)^{2\alpha}}.
\end{aligned}$$

Second,

$$\begin{aligned}
|b_k^{2,2,2}| &\leq C_0(T, \Delta t_0, |x|_E)\Delta t^2 \left(\mathbb{E}|\Delta t \sum_{\ell=0}^{k-1} AS_{\Delta t}^{k-\ell+1}\Psi_{\Delta t}(X_\ell)|_H^2\right)^{\frac{1}{2}} ds dt\\
&\leq C_0(T, \Delta t_0, |x|_E)\Delta t^2 \Delta t \sum_{\ell=0}^{k-1} \|A^{2\alpha} S_{\Delta t}^k\|_{\mathscr{L}(H)} \|A^{1-2\alpha} S_{\Delta t}\|_{\mathscr{L}(H)}\\
&\leq C(T, \Delta t_0, |x|_E)\Delta t^{1+2\alpha},
\end{aligned}$$

using that $\Delta t \sum_{\ell=0}^{k-1} \frac{1}{((k-\ell)\Delta t)^{2\alpha}} \leq C_\alpha(T) < \infty$ for $\alpha \in [0, \frac{1}{2})$.

It remains to treat $b_k^{2,2,3}$. Using the Malliavin calculus duality formula (6.2) and the chain rule,

$$b_k^{2,2,3} = \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \sum_{\ell=0}^{k-1} \int_{t_\ell}^{t_{\ell+1}} \sum_{j\in\mathbb{N}} \mathbb{E}\big[D^2u^{(\Delta t)}(T-t, \tilde{X}(t))$$
$$.\big(D\Psi_{\Delta t}(\tilde{X}(s)).(AS_{\Delta t}^{k-\ell+1}e_j), \mathscr{D}_r^{e_j}\tilde{X}(t)\big)\big]drdsdt$$
$$+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \sum_{\ell=0}^{k-1} \int_{t_\ell}^{t_{\ell+1}} \sum_{j\in\mathbb{N}} \mathbb{E}\big[\langle Du^{(\Delta t)}(T-t, \tilde{X}(t)), D^2\Psi_{\Delta t}(\tilde{X}(s))$$
$$.(AS_{\Delta t}^{k-\ell+1}e_j, \mathscr{D}_r^{e_j}\tilde{X}(s))\rangle\big]drdsdt.$$

Let $\eta \in (\frac{1}{4}, 1)$, which allows us to use inequality (2.2). Thanks to Theorems 4.1 and 4.2, Lemmas 6.1 and 6.4,

$$|b_k^{2,2,3}| \le \Delta t^2 \sum_{\ell=0}^{k-1} \sum_{j\in\mathbb{N}} \big(\Delta t C_0(T, \Delta t_0, |x|_E)$$
$$+ \int_{k\Delta t}^{(k+1)\Delta t} \frac{C_\eta(T, \Delta t_0, |x|_E)}{(T-t)^\eta} dt\big)|AS_{\Delta t}^{k-\ell+1}e_j|_H$$
$$\le C(T, \Delta t_0, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^\eta} dt\ \Delta t$$
$$\sum_{\ell=0}^{k-1} \|(-A)^{2\alpha}S_{\Delta t}^{k-\ell}\|_{\mathscr{L}(H)} \sum_{j\in\mathbb{N}} \frac{\lambda_j^{1-2\alpha}\Delta t}{1+\lambda_j\Delta t}$$
$$\le C_\alpha(T, \Delta t_0, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^\eta} dt\, \Delta t^{2\alpha}.$$

Finally,

$$|b_k^{2,2}| \le C_\alpha(T, \Delta t_0, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \big(1 + \frac{1}{t^{2\alpha}} + \frac{1}{(T-t)^{2\alpha}}\big)dt\ \Delta t^{2\alpha}.$$

Treatment of $b_k^{2,3}$

Applying Theorem 4.1 and Lemma 6.1 directly gives

$$|b_k^{2,3}| \le C_0(T, \Delta t_0, |x|_E)\Delta t^2.$$

Treatment of $b_k^{2,4}$

Using the Malliavin calculus duality formula (6.2) and the chain rule,

$$\begin{aligned}
b_k^{2,4} &= \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}\left[\sum_{j\in\mathbb{N}} \langle Du^{(\Delta t)}(T-t,\tilde{X}(t)), e_j\rangle \int_{k\Delta t}^{t} \langle D\Psi_{\Delta t}^{j}(\tilde{X}(s)), S_{\Delta t} dW(s)\rangle\right] dt \\
&= \int_{k\Delta t}^{(k+1)\Delta t} \int_{k\Delta t}^{t} \sum_{i,j\in\mathbb{N}} \mathbb{E}\big[D^2u^{(\Delta t)}(T-t,\tilde{X}(t)). \\
&\quad (e_j, \mathscr{D}_s^{e_i}\tilde{X}(t))\langle D\Psi_{\Delta t}^{j}(\tilde{X}(s)), S_{\Delta t} e_i\rangle\big] ds dt \\
&= \int_{k\Delta t}^{(k+1)\Delta t} \int_{k\Delta t}^{t} \sum_{i\in\mathbb{N}} \mathbb{E}\big[D^2u^{(\Delta t)}(T-t,\tilde{X}(t)).(D\Psi_{\Delta t}(\tilde{X}(s))S_{\Delta t}e_i, \mathscr{D}_s^{e_i}\tilde{X}(t))\big] ds dt \\
&= \int_{k\Delta t}^{(k+1)\Delta t} \int_{k\Delta t}^{t} \sum_{i\in\mathbb{N}} \mathbb{E}\big[D^2u^{(\Delta t)}(T-t,\tilde{X}(t)).(D\Psi_{\Delta t}(\tilde{X}(s))S_{\Delta t}e_i, S_{\Delta t}e_i)\big] ds dt,
\end{aligned}$$

indeed $\mathscr{D}_s\tilde{X}(t) = S_{\Delta t}$ for $k\Delta t \le s \le t \le (k+1)\Delta t$.

Let $\alpha \in (0, \frac{1}{2})$. Thanks to Theorem 4.2 and Lemma 6.1,

$$\begin{aligned}
|b_k^{2,4}| &\le C_{0,\alpha}(T, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^\alpha} dt \sum_{i\in\mathbb{N}} \frac{\Delta t}{\lambda_i^\alpha (1+\lambda_i \Delta t)^2} \\
&\le C_\alpha(T, \Delta t_0, |x|_E) \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{(T-t)^\alpha} dt \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1-\alpha}} \Delta t^{2\alpha}.
\end{aligned}$$

Conclusion

Gathering estimates above, for all $\alpha \in [0, \frac{1}{2})$,

$$\sum_{k=1}^{N-1} |b_k| \le C_\alpha(T, \Delta t_0, |x|_E) \Delta t^\alpha \sum_{k=1}^{N-1} \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{t^{\beta_1(\alpha)}(T-t)^{\beta_2(\alpha)}} dt,$$

with two parameters $\beta_1(\alpha), \beta_2(\alpha) \in [0, 1)$. Therefore

$$\sum_{k=1}^{N-1} \int_{k\Delta t}^{(k+1)\Delta t} \frac{1}{t^{\beta_1(\alpha)}(T-t)^{\beta_2(\alpha)}} dt \le \int_0^T \frac{1}{t^{\beta_1(\alpha)}(T-t)^{\beta_2(\alpha)}} dt < \infty.$$

This concludes the second part of the proof of Lemma 6.3.

6.2.4 Proof of Lemma 6.3, Part 3

It remains to treat $\sum_{k=1}^{N-1}|c_k|$. Let $\alpha\in(0,\frac{1}{2})$, and let $\varepsilon\in(\frac{1}{2}-\alpha,1-2\alpha)$ be an auxiliary parameter. Thanks to Theorem 4.2, and to Lemma 6.1, using $(-A)^{-\beta}e_j=\lambda_j^{-\beta}e_j$,

$$
\begin{aligned}
|c_k| &\leq C_{2\alpha+\varepsilon}(T,\Delta t_0,|x|_E)\int_{k\Delta t}^{(k+1)\Delta t}\frac{1}{(T-t)^{2\alpha+\varepsilon}}dt\sum_{j\in\mathbb{N}}\frac{\lambda_j^{-2\alpha-\varepsilon}\big(2\lambda_j\Delta t+(\lambda_j\Delta t)^2\big)}{(1+\lambda_j\Delta t)^2}\\
&\leq C_{2\alpha+\varepsilon}(T,\Delta t_0,|x|_E)\int_{k\Delta t}^{(k+1)\Delta t}\frac{1}{(T-t)^{2\alpha+\varepsilon}}dt\sum_{j\in\mathbb{N}}\frac{\Delta t^{\alpha}}{\lambda_j^{\alpha+\varepsilon}}\frac{2(\lambda_j\Delta t)^{1-\alpha}+(\lambda_j\Delta t)^{2-\alpha}}{(1+\lambda_j\Delta t)^2}.
\end{aligned}
$$

Note that $\sum_{j\in\mathbb{N}}\frac{1}{\lambda_j^{\alpha+\varepsilon}}<\infty$, since $\alpha+\varepsilon>\frac{1}{2}$. In addition, $\sup\limits_{z\geq 0}\frac{z^{\beta}}{(1+z)^2}<\infty$ for $\beta\in[0,2]$. Thus

$$\sum_{k=1}^{N-1}|c_k|\leq C_{\alpha}(T,\Delta t_0,|x|_E)\Delta t^{\alpha}.$$

6.2.5 Conclusion

Gathering the estimates on $\sum_{k=1}^{N-1}|a_k|$, $\sum_{k=1}^{N-1}|b_k|$ and $\sum_{k=1}^{N-1}|c_k|$, concludes the proof of Lemma 6.3.

6.3 Proof of Lemmas 6.1 and 6.4

The objective of this section is to provide the proofs of two auxiliary results used above: moment estimates for the numerical scheme, Lemma 6.1, and estimates for the Malliavin derivative, Lemma 6.4.

Proof of Lemma 6.1

Let $(\omega_n)_{n\in\mathbb{N}}$ be given by

$$\omega_n=\sum_{k=0}^{n-1}S_{\Delta t}^{n-k-1}\big(W((k+1)\Delta t)-W(k\Delta t)\big),$$

which solves $\omega_{n+1}=S_{\Delta t}\omega_n+S_{\Delta t}\big(W((n+1)\Delta t)-W(n\Delta t)\big)$.

Then (see [8]), there exists $C(T,\Delta t_0,M)\in(0,\infty)$ such that

$$\mathbb{E}\Big[\sup_{0\leq n\leq N}|X_n|_E^M+\sup_{0\leq n\leq N}|\omega_n|_E^M\Big]\leq C(T,\Delta t_0,M)(1+|x|_E)^M.$$

Recall that $AS_{\Delta t}=\frac{S_{\Delta t}-I}{\Delta t}$, and that $\|S_{\Delta t}\|_{\mathscr{L}(E)}\leq 1$.

For $n\Delta t \le t \le (n+1)\Delta t \le T$, using (6.1),

$$\begin{aligned}
&\big(\mathbb{E}|\tilde{X}(t)|_E^M\big)^{\frac{1}{M}}\\
&\le \big(\mathbb{E}|X_n|_E^M\big)^{\frac{1}{M}} + 2\frac{t-n\Delta t}{\Delta t}\big(\mathbb{E}|X_n|_E^M\big)^{\frac{1}{M}} + (t-n\Delta t)\big(\mathbb{E}|\Psi_{\Delta t}(X_n)|_E\big)^{\frac{1}{M}}\\
&\quad + \big(\mathbb{E}|S_{\Delta t}(W(t)-W(n\Delta t)|_E^M\big)^{\frac{1}{M}}\\
&\le C(1+|x|_E+|x|_E^3) + \frac{(t-n\Delta t)^{\frac{1}{2}}}{\Delta t^{\frac{1}{2}}}\big(\mathbb{E}|S_{\Delta t}(W((n+1)\Delta t)-W(n\Delta t)|_E^M\big)^{\frac{1}{M}},
\end{aligned}$$

and it remains to observe that $S_{\Delta t}(W((n+1)\Delta t) - W(n\Delta t) = \omega_{n+1} - S_{\Delta t}\omega_n$, and to use the estimate above.

This concludes the proof of Lemma 6.1. □

Proof of Lemma 6.4 Introduce the following notation: for every $s \in [0,T]$, $\ell_s = \lfloor \frac{s}{\Delta t} \rfloor \in \{0,\ldots,N\}$.

If $\ell_s \ge k$, then $s \ge k\Delta t$, and by definition of the Malliavin derivative, $\mathscr{D}_s X_k = 0$.

Note that $X_{\ell_s+1} = S_{\Delta t}\Phi_{\Delta t}(X_{\ell_s}) + S_{\Delta t}(W((\ell_s+1)\Delta t) - W(\ell_s\Delta t))$, thus one has $\mathscr{D}_s X_{\ell_s+1} = S_{\Delta t}$.

First, if $k \ge \ell_s + 1$, using the chain rule, for all $h \in H$,

$$\mathscr{D}_s^h X_{k+1} = \mathscr{D}_s^h\big(S_{\Delta t}\Phi_{\Delta t}(X_k)\big) = S_{\Delta t}\Phi_{\Delta t}'(X_k)\mathscr{D}_s^h X_k,$$

and since $\Phi_{\Delta t}$ is globally Lipschitz continuous, thanks to Lemma 2.1,

$$|\mathscr{D}_s^h(X_{k+1})|_H \le e^{\Delta t}|\mathscr{D}_s^h X_k| \le e^T|\mathscr{D}_s^h X_{\ell_s+1}|_H \le e^T|h|_H.$$

Second, for $0 \le s < k\Delta t \le t \le (k+1)\Delta t \le T$, and $h \in H$, using the chain rule and (6.1),

$$\begin{aligned}
|\mathscr{D}_s^h\tilde{X}(t)|_H &\le |\mathscr{D}_s^h\tilde{X}(t)|_H + (t-n\Delta t)|AS_{\Delta t}\mathscr{D}_s^h\tilde{X}(t)|_H\\
&\quad + (t-n\Delta t)|S_{\Delta t}\Psi_{\Delta t}'(X_k)\mathscr{D}_s^h X_k|_H\\
&\le (3+\Delta t|\Psi_{\Delta t}'(X_k)|_E)|\mathscr{D}_s^h X_k|_H,
\end{aligned}$$

and using the estimate above concludes the proof of Lemma 6.4. □

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