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Analysis of the Monte-Carlo error in a hybrid semi-lagrangian scheme

Charles-Edouard Bréhier[a] and Erwan Faou[b]
[a]*Institut de Mathématiques, Université de Neuchâtel, Rue Emile-Argand, Neuchâtel, Switzerland-2000*
[b]*INRIA & ENS Rennes - IRMAR, Université Rennes 1, Avenue Robert Schuman, F-35170 Bruz, France*

Abstract: We consider Monte-Carlo discretizations of partial differential equations based on a combination of semi-lagrangian schemes and probabilistic representations of the solutions. The goal of this paper is twofold. First we give rigorous convergence estimates for our algorithm: In a simple setting, we show that under an anti-CFL condition on the time-step δt and on the mesh size δx and for a reasonably large number of independent realizations N, we control the Monte-Carlo error by a term of order $\mathcal{O}(\sqrt{\delta t/N})$. Then, we show various applications of the numerical method in very general situations (nonlinear, different boundary conditions, higher dimension) and numerical examples showing that the theoretical bound obtained in the simple case seems to persist in more complex situations.

KEY WORDS semi-lagrangian methods, Monte-Carlo methods, reduction of variance

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1 Introduction

In this paper we consider numerical schemes combining the principle of semi-lagrangian and Monte-Carlo methods applied to various partial differential equations.

The main goals of this work are the following. First we present the method and show how it can be easily extended to various situations (linear, nonlinear, different boundary conditions). Then we state the main theoretical result of this paper, where we prove rigorously the convergence of the method together with an error estimate in a simple situation. By various numerical examples, we show that this error bound persists in more elaborated situations.

Before presenting the method, let us first consider a linear transport equation of the form

$$\partial_t u(t,x) = f(x)\cdot\nabla u(t,x), \quad x\in\mathbb{R}^d, \quad u(0,x)=u_0(x),$$

where u_0 is a given function. Under some regularity assumptions and existence of the flow associated with the vector field $f(x)$ in $\mathbb{R}^d$, the solution of this equation is given by the characteristics representation $u(t,x)=u(0,\varphi_t(x))$, where $\varphi_t(x)$ is the flow associated with the ordinary differential equation $\dot y = f(y)$ in $\mathbb{R}^d$. In this context, semi-lagrangian schemes can be described as follows. Let us consider a set of grid nodes x_j, $j\in K$ in $\mathbb{R}^d$ ($K=\mathbb{N}$ or a finite set) and an interpolant operator $\mathcal{I}$ mapping vectors of values at the nodes, $(u_j)\in\mathbb{R}^K$ to a function $(\mathcal{I}u)(x)$ defined over the whole domain. In this paper we will consider the case where $x_j=j(\delta x)$, $j\in\mathbb{Z}^d$, δx is the space mesh size, and $\mathcal{I}$ a standard linear interpolation operator. Given approximations u_j^n of the exact solution $u(t_n,x_j)$ at times $t_n=n(\delta t)$ and points x_j, the previous formula gives an approximation scheme for u_j^{n+1} obtained by solving the ordinary differential equation $\dot y=f(y)$ between t_n and t_{n+1}: $u_j^{n+1}=(\mathcal{I}u^n)(\Phi_{\delta t}(x_j))$, where Φ_h is the numerical flow associated with a time integrator.

These methods are particularly interesting when the vector field $f(x;u)$ depends on the solution u making the transport equation nonlinear, see for instance [12, 4, 11] and the references therein. This is the case when an advection term is present for instance, or for Vlasov equations (see for instance [2]). In these situations, standard semi-lagrangian schemes are based on solving equations of the form

$$\partial_t u(t,x) = f(x;u^n)\cdot\nabla u(t,x),$$

between t_n and t_{n+1}, where u^n denotes the solution at time t_n. In other words the vector field is frozen in u^n (in the language of geometric numerical integration, it is Crouch and Grossman method, see [1]). If moreover

*Correspondence to: E. Faou, Erwan.Faou@inria.fr

the vector field $f(x;u)$ possesses some geometric structure for all functions u, the numerical integrator can be chosen to preserve this structure (for example symplectic integrator in the Vlasov case).

In many situations, a diffusion term is present, and the equation can be written (in the linear case)

$$\partial_t u(t,x) = \tfrac{1}{2}\sum_{i,j=1}^d a_{i,j}(x)\partial_{x_i}\partial_{x_j}u(t,x) + \sum_{i=1}^d f_i(x)\partial_{x_i}u(t,x), \quad x\in\mathbb{R}^d, \quad u(0,x)=u_0(x), \tag{1.1}$$

where $\sigma(x)$ is a $d\times k$ matrix and $a(x)=\sigma(x)\sigma(x)^*$. Here and throughout the article we denote by σ^* the transpose of a matrix σ.

In this case the solution admits the probabilistic representation

$$u(t,x) = \mathbb{E}\, u_0(X_t^x),$$

where X_t^x is the stochastic process associated with the stochastic differential equation

$$\mathrm{d}X_t^x = f(X_t^x)\mathrm{d}t + \sigma(X_t^x)\mathrm{d}B_t, \quad X_0^x = x, \tag{1.2}$$

where $(B_t)_{t\geq 0}$ is a standard k-dimensional Brownian Motion.

In general, the law of the random variable X_t^x is not explicitly known, and we are not able to compute the expectation. The classical approximation procedures for such problems are Monte-Carlo methods: if we assume that we are able to compute N independent realizations $(X_t^{x,m})_{1\leq m\leq N}$ of the law of X_t^x, we can approach $u(t,x)$ with

$$\frac{1}{N}\sum_{m=1}^N u_0(X_t^{x,m}). \tag{1.3}$$

In general, the variance of the random variables $u_0(X_t^{x,m})$ is of size t and the law of large numbers ensures that the statistic error made is typically of order $\mathcal{O}(\sqrt{T/N})$ for an integration over the interval $[0,T]$. To this error must be added the error in the approximation of the process X_t^x by numerical schemes of Euler type for instance. This error is of order $\mathcal{O}(\delta t)$, see for instance [8, 10, 13] and the reference therein for analysis of the weak-error in the numerical approximation of stochastic differential equations. If a global knowledge of the solution is required, the above operation must be repeated for different values of x_j on the grid.

The numerical method we study in this paper (that was previously introduced in [3]) is based on the Markov property of the associated stochastic processes: we have for any x_j on the spatial grid and locally in time

$$u(t_{n+1},x_j) = \mathbb{E}u(t_n, X_{\delta t}^{x_j}), \tag{1.4}$$

which is the formula we aim at discretizing. Using the Euler method to compute a numerical approximation of $X_{\delta t}^{x_j}$, we end up with the following numerical scheme

$$u_j^{n+1} := \frac{1}{N}\sum_{m=1}^N \big(\mathcal{I}u^n\big)(x_j + \delta t f(x_j) + \sqrt{\delta t}\sigma(x_j)\mathcal{N}^{n,m,j}), \tag{1.5}$$

where the random variables $(\mathcal{N}^{n,m,j})_{1\leq m\leq M}$ are independent standard, k-dimensional Gaussian random variables. Note that the main difference between the standard Monte-Carlo method is that the average is computed at every time step.

The principle of using (random) characteristic curves in (1.4) over a time interval of size δt and to use an interpolation procedure to get functions defined on the whole domain fits in the semi-lagrangian framework. The addition of the Monte-Carlo approximation then justifies the use of the hybrid terminology.

As in the deterministic case described above, it is clear the method can be adapted to situations where the drift term $f(x)$ and the noise term $\sigma(x)$ depend on the solution u. In Section 3 we will show how this method can be easily extended to more general situations and give some numerical examples in nonlinear situations, and also in the case where the PDE (1.1) is complemented with different kind of boundary conditions (Full space, Dirichlet, Neumann). Indeed, representation formulae such as (1.4) still hold true in the case of periodic boundary conditions, Dirichlet or Neumann condition on bounded domains.

Naive calculations would suggest that the previous scheme approximates the exact solution with an error of size $\mathcal{O}(\delta t + \frac{\delta x^2}{\delta t} + \frac{1}{\sqrt{N}})$. However, numerical simulations indicate a better convergence rate, namely $\mathcal{O}(\delta t + \frac{\delta x^2}{\delta t} + \sqrt{\delta t/N})$. Surprisingly, it turned out to be very difficult to prove rigorously such bound in general situations, mainly because it is linked with the control of the long time behavior of products of random matrices.

Nevertheless, we prove the better convergence estimate in a simple situation, that is $f=0$, in dimension $d=1$, with $\sigma(x)=1$ and periodic boundary conditions on a domain $D=(0,1)\subset\mathbb{R}$ (see Theorems 2.2 and 2.3). This result is given in Section 2, and proved in Sections 4 and 5.

In Section 3, we analyze numerically several extensions of the scheme, in nonlinear situations (Burgers), in higher dimension, or with different kinds of boundary conditions (Dirichlet). As shown by the numerical results, the theoretical error estimates of Section 2 and the variance reduction phenomena persists in all these more complex cases.

2 A convergence result

The aim of this section is to state a rigorous convergence result in a simple case where a complete mathematical analysis is possible. We consider the linear heat equation (1.1) with a smooth initial condition and with periodic boundary conditions in dimension $d = 1$, with $f(x) = 0$ and $\sigma(x) = 1$. In this case, the solution of (1.1) is given by the representation formula $u(t, x) = \mathbb{E}u_0(x + B_t)$.

The goal of this Section is to analyze the effect of the discretization (1.5) in this situation where the exact solution can be easily computed.

We denote by $\mathcal{C}^\infty_{per}$ the space of periodic, $\mathcal{C}^\infty$ functions of period 1 defined on $\mathbb{R}$. We also define $\mathcal{C}^\infty_{per}((0,1))$ to be the space of the restrictions to the interval $[0,1]$ of functions in $\mathcal{C}^\infty_{per}$.

Finally, we define L^2 to be the space $L^2((0,1),\mathbb{R})$ of square-integrable functions on the interval $(0,1)$, and H^1 the closure of $\mathcal{C}^\infty_{per}((0,1))$ in the space $H^1((0,1),\mathbb{R})$.

Those spaces are endowed with the respective norm and semi-norm

$$\|f\|^2_{L^2} = \int_0^1 |f(x)|^2 \mathrm{d}x, \quad \text{and} \quad |f|^2_{H^1} = \int_0^1 |f'(x)|^2 \mathrm{d}x.$$

2.1 Interpolation operator

For a given integer $M_S \geq 1$, we discretize the space interval $(0,1)$ with the introduction of nodes $x_j = j\delta x$ for $j \in S := \{0, \dots, M_S - 1\}$, with the condition $x_{M_S} = M_S \delta x = 1$. We set $\mathrm{V}_S = \{(u_j)_{j\in S}\} \cong \mathbb{R}^{M_S}$ the set of functions defined on the points of the grid.

We use linear interpolation to reconstruct functions on the whole interval from values at the nodes. We define an appropriate basis made of periodic and piecewise linear functions for $k \in S = \{0, \dots, M_S - 1\}$. We set for $x \in [x_k - 1/2, x_k + 1/2]$,

$$\phi_k(x) = \hat{\phi}(\frac{x - x_k}{\delta x}), \quad \text{where} \quad \hat{\phi}(x) = \begin{cases} 0 \text{ if } |x| > 1, \\ 1 - |x| \text{ if } |x| \leq 1, \end{cases}$$

and extend the function ϕ_k by periodicity on $(0,1)$. Hence ϕ_k is a piecewise linear periodic function which satisfies $\phi_k(x_j) = \delta_{kj}$ the Kronecker symbol. Note that we have $\sum_{k\in S} \phi_k(x) = 1$ for all $x \in (0,1)$.

We define the following projection and interpolation operators

$$\mathcal{P} : \begin{cases} H^1 \to \mathrm{V}_S \\ f \mapsto (f(x_j))_{j\in S} \end{cases} \quad \text{and} \quad \mathcal{I} : \begin{cases} \mathrm{V}_S \to H^1 \\ u = (u_k)_{k\in S} \mapsto \sum_{k=0}^{M_S-1} u_k \phi_k(x). \end{cases}$$

Clearly, $\mathcal{P} \circ \mathcal{I}$ is the identity on V_S; nevertheless the distance between the identity and the composition of the operators $\mathcal{I} \circ \mathcal{P}$ depends on the functional spaces and on the norms. Below, we give the estimates that are useful in our setting. Notice the identity $\mathcal{I} \circ \mathcal{P}(\mathbb{1}) = \mathbb{1}$.

2.2 Discrete norms

For any $u = (u_j)_{j\in S} \in \mathrm{V}_S$ we define the discrete ℓ^2 norm and h^1 semi-norm as

$$\|u\|^2_{\ell^2} = \delta x \sum_{j\in S} u_j^2 \quad \text{and} \quad |u|^2_{h^1} = \delta x \sum_{j\in S} \frac{(u_{j+1} - u_j)^2}{\delta x^2},$$

where we use the extension by periodicity of the sequence (u_j) for the definition of the h^1 semi-norm: we thus have $u_{M_S} = u_0$. We also define a norm with $\|u\|_{h^1} = (\|u\|^2_{\ell^2} + |u|^2_{h^1})^{1/2}$.

With these notations, we have the following classical approximation results, whose proof is given in appendix

Proposition 2.1. *There exists a constant $c > 0$ such that for any mesh size $\delta x = 1/M_S$, and any sequence $u = (u_j) \in V_S$ we have:*

$$|u|_{h^1} = |\mathcal{I}u|_{H^1}, \quad \textit{and} \quad \|u\|^2_{\ell^2} = \|\mathcal{I}u\|^2_{L^2} + c\delta x^2 |u|^2_{h^1}.$$

Moreover, for any function $f \in H^1$ we have

$$\|f - (\mathcal{I} \circ \mathcal{P})f\|^2_{L^2} \leq c\delta x^2 \left(|f|^2_{H^1} + |(\mathcal{I} \circ \mathcal{P})f|^2_{H^1} \right).$$

2.3 Numerical method

We consider a final time $T > 0$, and an integer M_T, such that we divide the interval $[0, T]$ into M_T intervals of size $\delta t := \frac{T}{M_T}$. With these notations, the numerical scheme (1.5) is the application $u^n \mapsto u^{n+1}$ from V_S to itself written

$$u_j^{n+1} = \frac{1}{N} \sum_{m=1}^{N} \left(\sum_{k \in S} u_k^n \phi_k(x_j + \sqrt{\delta t} \mathcal{N}^{n,m,j}) \right), \tag{2.1}$$

where the random variables $\mathcal{N}^{n,m,j}$ are defined as follows: for any $n \in \{0, \dots, M_T - 1\}$

$$\mathcal{N}^{n,m,j} = \frac{B^{(m,j)}_{(n+1)\delta t} - B^{(m,j)}_{n\delta t}}{\sqrt{\delta t}}, \tag{2.2}$$

for some independent Brownian Motions $(B^{(m,j)})$ for $1 \leq m \leq N$ and $0 \leq j \leq M_S - 1$.

Therefore the random variables $\mathcal{N}^{n,m,j}$ for $0 \leq n \leq M_T - 1$, $1 \leq m \leq N$ and $j \in S$ are independent standard normal variables.

We start with an initial condition $u^0 = (u_k^0 = u_0(x_k))$, which contains the values of the initial condition at the nodes. To obtain simple expressions with products of matrices, we consider that vectors like u^0 are column vectors.

We then define the important auxiliary sequence $v^n \in V$ satisfying the following relations:

$$\begin{aligned} v_j^{n+1} &= \frac{1}{N} \sum_{m=1}^{N} \left(\sum_{k \in S} v_k^n \mathbb{E}[\phi_k(x_j + \sqrt{\delta t} \mathcal{N}^{n,m,j})] \right) \\ &= \sum_{k \in S} v_k^n \mathbb{E}[\phi_k(x_j + \sqrt{\delta t} \mathcal{N}^{n,1,j})], \end{aligned} \tag{2.3}$$

with the initial condition $v^0 = u^0$. Indeed, for any $0 \leq n \leq M_T$ the vector v^n is the expected value - defined component-wise - of the random vector u^n.

2.4 Main result

With the previous notations, we have the following error estimate:

Theorem 2.2. *Assume that the initial condition u_0 is of class $\mathcal{C}^2$. For any $p \in \mathbb{N}$ and any final time $T > 0$, there exists a constant $C_p > 0$, such that for any $\delta t > 0$, $\delta x > 0$ and $N \in \mathbb{N}^*$ we have for any $n \in \mathbb{N}$ with $n\delta t \leq T$*

$$\sup_{j \in S} |u(t_n, x_j) - v_j^n| \leq C_0 \frac{\delta x^2}{\delta t} \sup_{x \in [0,1]} |u_0''(x)| \tag{2.4}$$

and

$$\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 \leq C_p |u^0|_{h^1}^2 (1 + \frac{\delta x^2}{\delta t}) \left(1 + \frac{\delta x}{\delta t} + \frac{\delta x^2}{\delta t^2} (1 + |\log(\delta t)|) \right)^p \left(\frac{\delta t}{N} + \frac{1}{N^{p+1}} \right). \tag{2.5}$$

The control of the first part of the error is rather classical, while the estimate on the Monte-Carlo error given by (2.5) is more original and requires more attention in its analysis and in its proof.

First, we observe that the estimate is only interesting if a condition of anti-CFL type is satisfied: for some constant $c > 0$ we require

$$\frac{\delta x}{\delta t} \max(1, \sqrt{|\log(\delta t)|}) < c.$$

Such a control of the dependence between the discretization steps δt and δx is the appropriate condition for semi-lagrangian methods; the inequality is reversed for Courant-Friedrichs-Lewy conditions in finite difference schemes - it writes $\frac{\delta t}{\delta x^2} < c$ for a finite difference scheme in the case of PDEs like (1.1).

We then identify in (2.5) a leading term of size $\frac{\delta t}{N}$, which corresponds to the statistical error in a Monte-Carlo method for random variables of variance δt, and a remaining term, which goes to 0 with arbitrary order of convergence with respect to the number of realizations N. This second term is obtained via a bootstrap argument. Indeed it is easy to get the classical estimate with $p = 0$. The core of the proof is contained in the recursion which allows to increase the order from p to $p + 1$; it heavily relies on the spatial structure of the noise and on the choice of the ℓ^2-norm.

Thanks to (2.5) when $p = 1$, we see that interpreting Theorem 2.2 as a reduction of variance – when we compare our method with a Monte-Carlo approximation which is performed to compute separately the value of the solution at the final time for different grid points – with a size δt is valid: we bound the error with

$$\frac{\delta t}{N} + \frac{1}{N^2} \leq \frac{\delta t^2}{2} + \frac{3}{2N^2},$$

which can be compared with a classical Monte-Carlo bound with the variance δt: we have for any sample $(Y_1, \dots, Y_N)$ of a random variable Y

$$\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right) = \frac{\mathrm{Var}(Y)}{N} \leq \frac{\mathrm{Var}(Y)^2}{2} + \frac{1}{2N^2}.$$

Another way of controlling the Monte-Carlo error is given in the following estimate:

Theorem 2.3. *Assume that the initial condition u_0 is of class $\mathcal{C}^2$. Then for any final time $T > 0$, there exists a constant $C > 0$, such that for any $\delta t > 0$, $\delta x > 0$ and $N \in \mathbb{N}^*$ we have for any $n \in \mathbb{N}$ with $n\delta t \leq T$*

$$\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 \leq 2C|u^0|_{h^1}^2(1 + \frac{\delta x^2}{\delta t})\frac{\delta t}{N}$$

whenever N is sufficiently large:

$$\frac{C}{N}(1 + \frac{\delta x}{\delta t} + \frac{\delta x^2}{\delta t^2}(1 + |\log(\delta t)|)) \leq \frac{1}{2}.$$

Despite the complexity of the estimates, as explained in the introduction we can interpret them as giving an upper bound of size $\mathrm{O}(\frac{\sqrt{\delta t}}{\sqrt{N}})$ for the Monte-Carlo error.

The control of the Monte-Carlo error in Theorem 2.2 relies on several arguments. Firstly, the first factor corresponds to the accumulation of the variances appearing at each time step - where two sources of error are identified: the random variables involve a stochastic diffusion process evaluated at time δt, and an error is introduced by the interpolation procedure. To obtain another factor, we observe that the independence of the random variables appearing for different nodes implies that only diagonal entries of some matrices appear - see (5.12). However, this independence property also complicates the proof: the solutions are badly controlled with respect to the h^1 semi-norm. We then propose a decomposition of the error where the number of realizations N appears in the variance with the different orders 1 and 2: the first part is controlled by δt and δx, while the second one is only bounded. We finally use recursively this decomposition in order to improve the estimate, with a bootstrap argument; alternatively at this stage we prove Theorem 2.3.

Theorems 2.2 and 2.3 are proved in Section 5.

3 Numerical results and extensions

In this Section, we provide numerical simulations obtained with the Monte-Carlo semi-lagrangian method in different situations. One the one hand, the variance estimates of Theorems 2.2 and 2.3 seem to be more general than only for the one-dimensional, periodic, linear case, for which we proved the result. On the other hand, we want to show how to adapt the method for instance in the case of the Burgers equation.

We start with an illustration of Theorem 2.2, where we recover the expected order of convergence for the variance see Figure 2. We then move from periodic to homogeneous Dirichlet boundary conditions, with an important discussion on the problem of the approximation of the process killed at the boundary; we numerically recover the same order of convergence as for the periodic setting.

We then perform a test in dimension 2, in the linear, periodic case. Our numerical investigation provides again convergence at order 1/2 with respect to δt, in the same regime of parameters where $\delta t = \delta x$, and with a discrete ℓ^2-norm adapted to the two-dimensional setting.

We justify the choice of the regime $\delta t = \delta x$ with the two following arguments. On the one hand, in general in (2.4) $\frac{\delta x^2}{\delta t}$ must be replaced with $\delta t + \frac{\delta x^2}{\delta t}$, which is minimal when choosing $\delta t = \delta x$. On the other hand, this choice is dictated - up to a logarithmic factor - by the anti-CFL condition $\frac{\delta x}{\delta t}\max(1, \sqrt{|\log(\delta t)|}) < c$; this condition allows to control the factor in (2.5) which is evaluated at power p.

Finally, we consider a non-linear example: the viscous Burgers equation. We consider homogeneous Dirichlet boundary conditions. We first show how to adapt the scheme to this situation. Then we show numerical simulations to analyze the rate of convergence in dimension 1, and finally in dimension 2 we represent the behavior of the solution in Figures 6 and 7 with one realization of the scheme.

3.1 Illustration of Theorem 2.2

The first numerical example we consider is a simulation of the solution of the heat equation in the spatial domain $(0,1)$ in periodic setting. We introduce the viscosity parameter ν so that the problem is

$$\frac{\partial u(t,x)}{\partial t} = \nu \frac{\partial^2 u(t,x)}{\partial x^2}, \text{ for } t>0, x\in(0,1), \quad u(0,x)=u_0(x) \text{ for } x\in(0,1), \tag{3.1}$$

with the boundary condition $u(t,1)=u(t,0)$ for $t\geq 0$. For the numerical simulation of Figure 1, we choose $\nu=0.01$, and $u_0(x)=\sin(2\pi x)$. The exact solution satisfies $u(t,x)=\exp(-4\pi^2\nu t)\sin(2\pi x)$. The discretization parameters are $\delta t=\delta x=0.01$ and $N=100$.

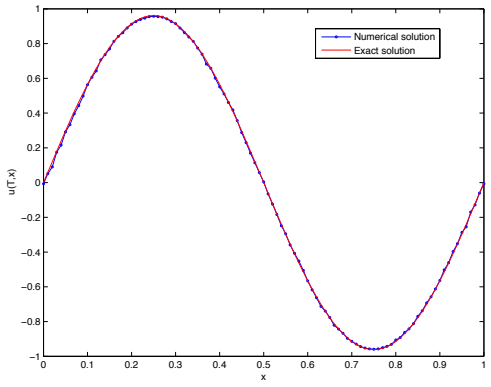


Figure 1: Solution at time $T=0.1$ with $\delta t=\delta x=1/N=0.01$

The bound of Theorem 2.2 is illustrated with Figure 2, where we represent the error in logarithmic scales for different values of the parameters.

We study the convergence of the scheme, with a numerical simulation which confirms the order of convergence with respect to the parameters $\delta t=\delta x$ of the Monte-Carlo error. The final time is $T=0.1$, the viscosity is $\nu=0.1$ and the initial condition is $u_0(x)=\cos(2\pi x)$. We compare the numerical solution u^n with the exact solution; with our set of parameters, we manage to only observe the Monte-Carlo error, which in the regime we consider is larger than the other error terms in the estimates of Theorem 2.2. The mean-square error in the ℓ^2 norm is estimated with a sample of size 20.

The error in Figure 2 is represented in logarithmic scales. The parameters δt and δx are equal and satisfy $\delta t=\delta x=\frac{1}{n}$ for the following values $n=50,100,200,400,800,1600,3200$. Each line is obtained when we draw the logarithm of the Error as a function of $\log_{10}(n)$, for a fixed value of $N\in\{10,20,40,80\}$. The dot-line represents a straight-line with slope $-1/2$.

This experiment confirms that the Monte-Carlo error is of order $1/2$ with respect to the parameters when $\delta t=\delta x$, as (2.5) claims. In addition, notice that the shift between the lines when N varies corresponds to the size $1/\sqrt{N}$ of the Monte-Carlo error.

3.2 The method for Dirichlet boundary conditions

Now, we would like to show how it is possible to adapt our method in the case of Dirichlet boundary conditions. Let us consider the equation (3.1), but with boundary conditions $u(t,x)=0$ for $t>0$ and $x\in\partial D=\{0,1\}$. The representation formula then involves the family of the first-exit times of the process $X_t^x=x+\sqrt{\nu}B_t$ starting from the different points of the domain: If we define $\tau^x=\inf\{t>0; X_t^x\in D^c\}$, then the solution satisfies

$$u(t,x)=\mathbb{E}\left[u_0(X_t^x)\mathbb{1}_{t\leq\tau^x}\right]; \tag{3.2}$$

the stochastic process is killed when it reaches the boundary. Note that this formula extends to more general PDE of the form (1.1) with the associated process (1.2).

The numerical approximation becomes more complicated, since we also need an accurate approximation of the stopping times. This problem is well-known, and solutions have been proposed in [6] and [9] for the computation of (3.2) at a given point x using time discretization of the stochastic process X_t^x. We implement them in an efficient way thanks to the flexibility of the method.

In our case, we take advantage of the semi-lagrangian context to do a refinement near the boundary: for a discretization between the times t_n and t_{n+1}, we introduce a decomposition of the domain into an "interior"

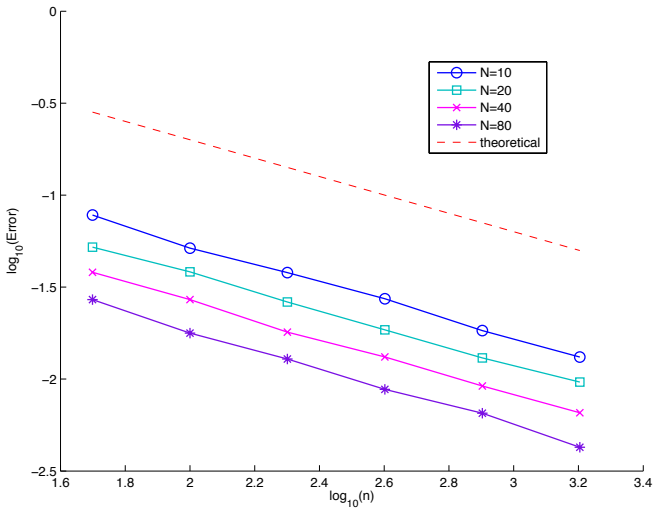


Figure 2: Error for periodic boundary conditions when $\delta t = \delta x = 1/n$, in logarithmic scales.

zone and a "boundary" zone, with different treatments. In the boundary zone, we refine in time and use a subdivision of $[n\delta t, (n+1)\delta t]$ of mesh size $\tau \leq \delta t$ and we use a possibly different value N_b for the number of Monte-Carlo realizations. Moreover, following [6] and [9], we introduce an exit test in the boundary zone, based on the knowledge of the law of exit of the diffusion process. The refinement allows us to get better result both quantitatively and qualitatively.

In the interior part, less computational effort is necessary and we can take $\tau = \delta t$ and $N_i < N_b$ for the size of the sample.

We give in Figure 3 the result of investigations on the convergence of the method when Dirichlet boundary conditions are applied. We draw in logarithmic scales the error in terms of $n = 1/\delta t = 1/\delta x$, with $n = 50, 100, 200, 400, 800$, with different values of the Monte-Carlo parameter $N_i = 10, 20, 40, 80$. We have chosen on the interval $(-1, 1)$ the initial function $u_0(x) = \sin(\pi \frac{x+1}{2})$, with the viscosity $\nu = 0.1$. The boundary zone is made of the intervals $(-1, -0.9)$ and $(0.9, 1)$, where we take $\tau = \delta t/10$ and $N_b = 10N_i$. The solutions are computed until time $T = 0.1$. Like in the case of periodic boundary conditions, the statistical error is dominant with respect to the other error terms; we compare with the exact solution, and to estimate the variance we use a sample of size 100.

The observation of Figure 3 shows that the Monte-Carlo error depends on the parameter $\delta t = \delta x$; the comparison with the "theoretical" line with slope $-1/2$ indicates a conjecture that the error is also of order $1/2$, like for the periodic case. The shift between the curves for different values of N corresponds in the error to a factor $1/\sqrt{N}$.

3.3 A test in dimension 2

We now perform a test for the heat equation in dimension 2, with periodic boundary conditions, in the domain $D = (0, 1)^2$:

$$\frac{\partial u(t,x,y)}{\partial t} = \nu \frac{\partial^2 u(t,x,y)}{\partial x^2} + \nu \frac{\partial^2 u(t,x,y)}{\partial y^2}, \text{ for } t > 0, x \in (0,1), \quad u(0,x,y) = u_0(x,y) \text{ for } x \in (0,1), \tag{3.3}$$

with periodic boundary conditions.

For our numerical tests, we have chosen $\nu = 0.1$, $u_0(x, y) = \sin(2\pi x)\sin(2\pi y)$, and we look at the approximation of the exact solution at time $T = 0.1$: $u(T, x, y) = \exp(-2\nu 4\pi^2 t)u_0(x, y)$.

The mean-square error is estimated with a sample of size 10.

The error is evaluated in the following discrete ℓ^2-norm:

$$\|u\|_{\ell^2} = \Big(\delta x^2 \sum_{j,k \in S} u_{j,k}^2\Big)^{1/2},$$

where $u_{j,k}$ is the approximation of the value of the solution at point $(j\delta x, k\delta x)$. We have normalized in such a way that if u is identically 1, then $\|u\|_{\ell^2} = 1$.

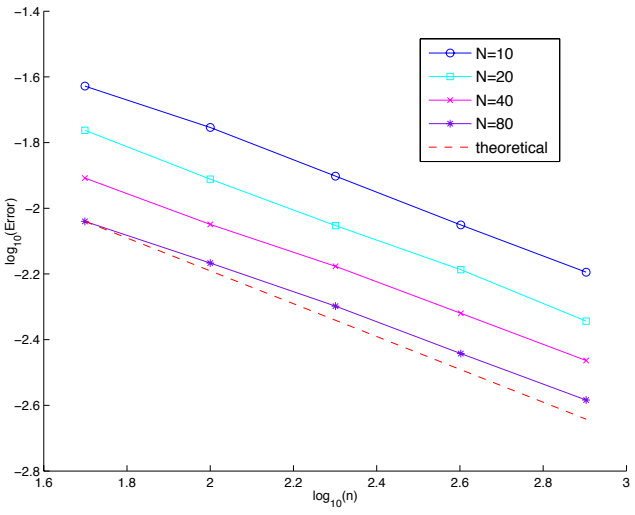


Figure 3: Error for Dirichlet boundary conditions when $\delta t = \delta x = 1/n$, in logarithmic scales.

In Figure 4, we represent the error in logarithmic scales. We have taken values in the following sets: $\delta t = \delta x = 1/n$ with $n \in \{100, 200, 400, 800, 1600\}$ and $N \in \{10, 20, 40, 80\}$.

We recover the same kind of result as for the two previous one-dimensional cases.

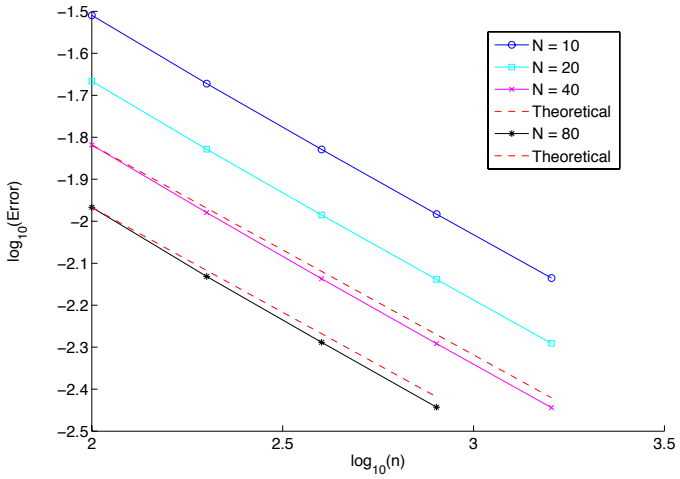


Figure 4: Error for the linear 2D case when $\delta t = \delta x = 1/n$, in logarithmic scales.

3.4 The method for some non-linear PDEs

We present a simple method to obtain approximations of the solution of the viscous Burgers equation in dimensions $d = 1$ and $d = 2$

$$\frac{\partial u}{\partial t} + (u.\nabla)u = \nu \Delta u + f.$$

It is defined on the domain $(-1, 1)^d$, with homogeneous Dirichlet boundary conditions - periodic ones would also have been possible. Compared with the situations described so far, we add a forcing term f, which may depend on time t, position x and the solution u.

As explained in the Introduction, we construct approximations u^n of the solution at discrete times $n\delta t$, introducing functions v^n such that for any $n \geq 0$ with the following semi-implicit scheme:

$$\frac{\partial v^{n+1}}{\partial t} + (u^n.\nabla)v^{n+1} = \nu \Delta v^{n+1} + f^n, \tag{3.4}$$

for any time $n\delta t \le t \le (n+1)\delta t$ and $x \in D$. The initial condition is $v^{n+1}(n\delta t, .) = u^n = v^n(n\delta t, .)$. The discrete-time approximation then satisfies $u^0 = u_0$ and $u^n = v^n(n\delta t, .)$. The forcing term here satisfies $f^n(t,x) = f(n\delta t, x, u^n(x))$.

On each subinterval $[n\delta t, (n+1)\delta t]$, we have

$$v^{n+1}(t,x) = \mathbb{E}[v^{n+1}(n\delta t, X_t^x)\mathbb{1}_{t<\tau^x} + \int_{n\delta t}^{t\wedge\tau^x} f^n(X_s^x)ds],$$

where the diffusion process X satisfies

$$dX_t^x = -u^n(X_t^x)dt + \sqrt{2\nu}dB_t, X_{n\delta t}^x = x.$$

The stopping times τ^x represents the first exit time of the process in the time interval $[n\delta t, (n+1)\delta t]$. Since $v^{n+1}(n\delta t, .) = u^n$, the scheme only requires the knowledge of the approximations u^n.

We first show numerical results in dimension $d = 1$. We look at the error at time $T = 0.1$, evaluated in the discrete ℓ^2-norm. Here $\nu = 0.1$, and the mean-square error is estimated with a sample of size 100. The initial condition satisfies $u_0(x) = \frac{\nu\pi \sin(\pi(x+1)/2)}{(2+cos(\pi(x+1)/2)}$; thanks to the Cole-Hopf transform, we have an explicit expression for the solution at any time $t \in [0,T]$: $u(t,x) = \frac{\nu\pi e^{-\nu\pi^2 T/4}\sin(\pi*(x+1)/2)}{(2+e^{-\nu\pi^2 T/4}cos(\pi(x+1)/2)}$, which makes the comparison simple. Parameters to get Figure 5 are the following: $\delta t = \delta x \in \{100, 200, 400, 800, 1600, 3200, 6400, 12800\}$ and $N \in \{10, 20, 40, 80, 160\}$.

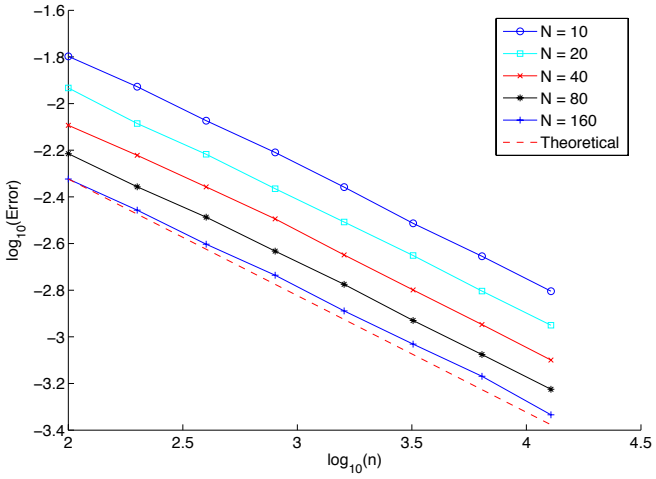


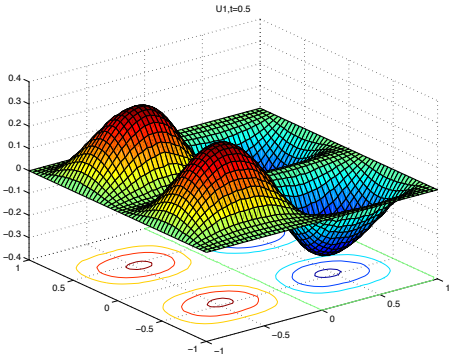
Figure 5: Error for the 1D Burgers equation $\delta t = \delta x = 1/n$, in logarithmic scales.

For the numerical simulations in dimension $d = 2$, we take the initial condition to be 0, and the forcing is $f(t,x) = (-\sin(\pi t)\sin(\pi x)\sin(\pi y)^2, -\sin(\pi t)\sin(\pi x)^2\sin(\pi y))$. The viscosity parameter is $\nu = 0.001$. The time step satisfies $\delta t = 0.02$, and the spatial mesh size is $\delta x = 0.04$. The "interior" zone is $(-0.8, +0.8)^2$, where $N_i = 10$; on the "boundary" zone, we have $N_b = 100 = 10N_i$, and $\tau = 0.002 = \delta t/10$.

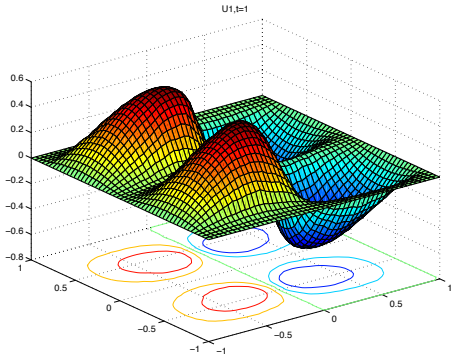
Both components of the velocity field u are represented in Figures 6 and 7 below at different times $t = 0.5, 1, 1.5, 2$.

4 Matrix formulation of the numerical scheme

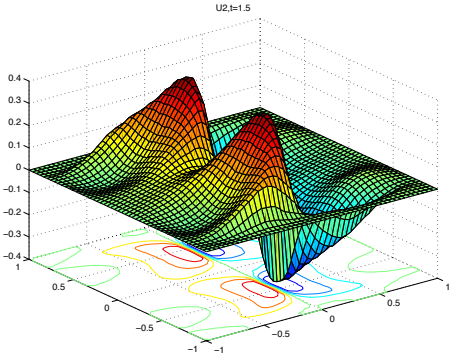
We now start the proof of Theorems 2.2 and 2.3, and we thus only focus on the case of the heat equation in dimension 1 and with periodic boundary conditions. We start with the definition and the study of important objects that are necessary to analyze the Monte-Carlo error. The fact that the equation is linear enables to rewrite the numerical solution in an easy way, with the definition of appropriate (random) matrices. We then give the main properties of these matrices. In particular, the choice of the norms from Section 2.2 appears to be essential at several points here; to treat other situations, it seems that one should introduce other appropriate norms.



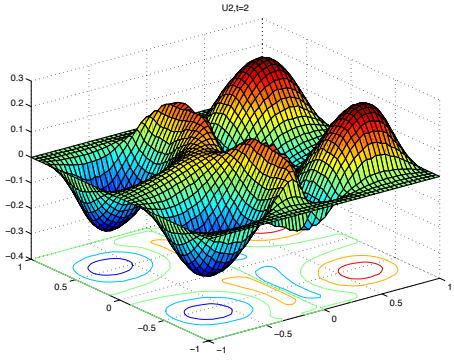
(a) $u_1, t = 0.5$



(b) $u_1, t = 1$

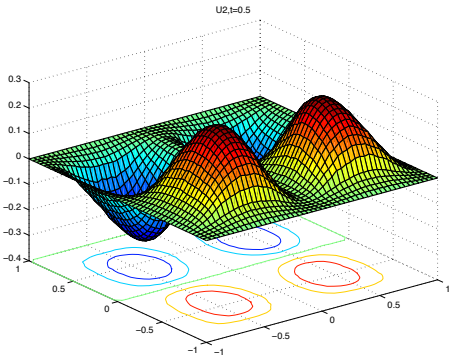


(c) $u_1, t = 1.5$

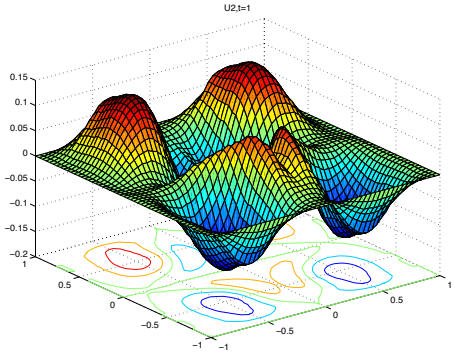


(d) $u_1, t = 2$

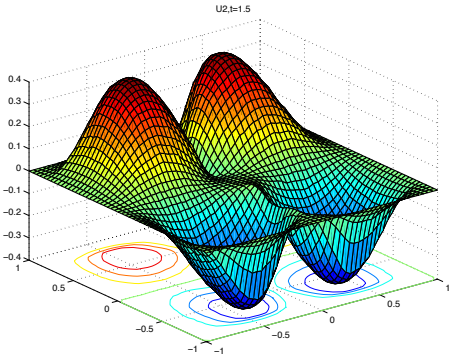
Figure 6: Solution of the 2D Burgers equation at different times - first component



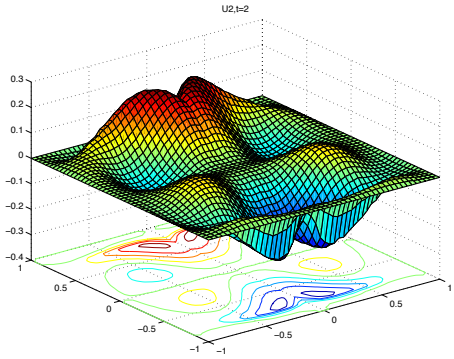
(a) $u_2, t = 0.5$



(b) $u_2, t = 1$



(c) $u_2, t = 1.5$



(d) $u_2, t = 2$

Figure 7: Solution of the 2D Burgers equation at different times - second component

The recursion (2.1) which defines the scheme can be rewritten with matrix notations: for column vectors of size M_S such that $(u^n)_j = u_j^n$, we see that

$$u^{n+1} = P^{(n)} u^n, \tag{4.1}$$

where the entries of square matrix satisfy for any $1 \leq j, k \leq M_S$

$$P_{j,k}^{(n)} = \frac{1}{N} \sum_{m=1}^{N} \phi_k(x_j + \sqrt{\delta t}\mathcal{N}^{n,m,j}). \tag{4.2}$$

Moreover we decompose these matrices into N independent parts: for $1 \leq m \leq N$

$$P^{(n)} = \frac{1}{N} \sum_{m=1}^{N} P^{(n,m)}, \tag{4.3}$$

with the entries $(P^{(n,m)})_{j,k} = \phi_k(x_j + \sqrt{\delta t}\mathcal{N}^{n,m,j})$.

We observe that the matrices $P^{(n,m)}$ are independent. Moreover in each one, for fixed n and m, the rows are independent, however in a row indexed by j two different entries are never independent, since they depend on the same random variable $\mathcal{N}^{n,m,j}$; moreover, the sum of coefficients in a row is 1. Since we also have $P_{j,k}^{(n,m)} \geq 0$, the matrices $P^{(n,m)}$ are (random) *stochastic* matrices - see Proposition 4.1 below.

All matrices $P^{(n,m)}$ have the same law; we define a matrix $Q = \mathbb{E}P^{(n,m)} = \mathbb{E}P^{(n)}$, by taking the expectations of each entry: for any $j, k \in S$

$$Q_{j,k} = \mathbb{E}[\phi_k(x_j + \sqrt{\delta t}\mathcal{N}^{n,m,j})]. \tag{4.4}$$

The right-hand side above does not depend on n, m since we take expectation. It only depends on j through the position x_j, not through the random variable $\mathcal{N}^{n,m,j}$. With these notations, the vectors v^n satisfy the relation $v^{n+1} = Qv^n$ - see (2.3) - and we have for any $n \geq 0$

$$u^n = \prod_{i=0}^{n-1} P^{(i)} u^0 = P^{(n-1)} \dots P^{(0)} u^0, \quad \text{and} \quad v^n = Q^n u^0. \tag{4.5}$$

We now state a few basic properties of the matrices $P^{(n,m)}$, $P^{(n)}$ and Q. First, we show that they are stochastic matrices. Second, we control their behavior with respect to the discrete norms and semi-norms. In order to prove the convergence result, we need other more technical properties which are developed during the proof.

Proposition 4.1. *For any $0 \leq n \leq M_T - 1$ and for any $1 \leq m \leq N$, almost surely $P^{(n,m)}$ is a stochastic matrix: for any indices $j, k \in S$ we have $P_{j,k}^{(n,m)} \geq 0$, and for any $j \in S$ we have $\sum_{k \in S} P_{j,k}^{(n,m)} = 1$.*

For any $0 \leq n \leq M_T - 1$, $P^{(n)}$ is also a random stochastic matrix.

The matrix Q is stochastic and symmetric - and therefore is bistochastic.

Proof. The stochasticity of the random matrices $P^{(n,m)}$ is a simple consequence of their definition (4.2) and of the relations $\phi_k(x) \geq 0$ and $\sum_{k \in S} \phi_k(x) = 1$. Since $P^{(n)}$ is a convex sum of the $P^{(n,m)}$, the property for those matrices also holds.

Finally, by taking expectation Q is obviously stochastic; symmetry is a consequence of (4.4), and of the property $\phi_{k+1}(x) = \phi_k(x - \delta x)$:

$$\begin{aligned} Q_{j,k} &= \mathbb{E}[\phi_k(x_j + \sqrt{\delta t}\mathcal{N}^{n,m,j})] = \mathbb{E}[\phi_0(x_j - x_k + \sqrt{\delta t}\mathcal{N}^{n,m,j})] \\ &= \mathbb{E}[\phi_0(x_k - x_j - \sqrt{\delta t}\mathcal{N}^{n,m,j})] = \mathbb{E}[\phi_0(x_k - x_j + \sqrt{\delta t}\mathcal{N}^{n,m,k})] = Q_{k,j}, \end{aligned}$$

since ϕ_0 is an even function, and since the law of $\mathcal{N}^{n,m,j}$ is symmetric and does not depend on j. However this symmetry property is not satisfied by the P-matrices, because the trajectories of these random variables are different when j changes. □

Thanks to the chain of equalities in the proof above, we see that $Q_{j,k}$ only depends on $k - j$, but we observe that no similar property holds for the matrices $P^{(n,m)}$.

We now focus on the behavior of the matrices with respect to the ℓ^2-norm. The following proposition is a discrete counterpart of the decreasing of the L^2-norm of solutions of the heat equation.

Proposition 4.2. *For any $0 \leq n \leq M_T - 1$ and for any $1 \leq m \leq N$, and for any $u \in V$ we have*

$$\mathbb{E}\|P^{(n,m)}u\|_{\ell^2}^2 \leq \|u\|_{\ell^2}^2 \quad \text{and} \quad \mathbb{E}\|P^{(n)}u\|_{\ell^2}^2 \leq \|u\|_{\ell^2}^2.$$

Proof. According to the definitions above (4.1) and (4.3), we have for any index j $(P^{(n,m)}u)_j = \sum_{k\in S} P^{(n,m)}_{j,k} u_k$. Thanks to the previous Proposition 4.1, we use the Jensen inequality to get

$$\begin{aligned}\mathbb{E}\|P^{(n,m)}u\|^2_{\ell^2} &= \delta x \sum_{j\in S} \mathbb{E}|(P^{(n,m)}u)_j|^2 \\ &\leq \delta x \sum_{j\in S}\sum_{k\in S} \mathbb{E}P^{(n,m)}_{j,k}|u_k|^2 \leq \delta x \sum_{k\in S}\left(\sum_{j\in S} Q_{j,k}\right)|u_k|^2;\end{aligned}$$

now we use the properties of the matrix Q - it is a bistochastic matrix according to Proposition 4.1 - to conclude the proof, since $\sum_{j\in S} Q_{j,k} = 1$. The extension to the matrices $P^{(n)}$ is straightforward. □

The matrix Q satisfies the same decreasing property in the ℓ^2-norm; moreover we easily obtain a bound relative to the h^1-semi norm:

Proposition 4.3. *For any $u \in V$, we have $\|Qu\|_{\ell^2} \leq \|u\|_{\ell^2}$ and $|Qu|_{h^1} \leq |u|_{h^1}$.*

Proof. The proof of the first inequality is similar to the previous situation for the random matrices. To get the second one, it suffices to define a sequence $\tilde{u}$ such that for any $0 \leq j \leq M_S - 1$ we have $\tilde{u}_j = \frac{u_{j+1}-u_j}{\delta x}$ - with the convention $u_{M_S} = u_0$. Then thanks to the properties of Q we have $\widetilde{Qu} = Q\tilde{u}$: for any $j \in S$

$$\begin{aligned}(\delta x)\widetilde{Qu}_j = (Qu)_{j+1} - (Qu)_j &= \sum_{k\in S} Q_{j+1,k}u_k - \sum_{k\in S} Q_{j,k}u_k \\ &= \sum_{k\in S} Q_{j,k-1}u_k - \sum_{k\in S} Q_{j,k}u_k = \sum_{k\in S} Q_{j,k}(u_{k+1}-u_k) = \delta x (Q\tilde{u})_j,\end{aligned}$$

using a translation of indices with periodic conditions, and the equality $Q_{j+1,k} = Q_{j,k-1}$ as explained above. As a consequence, we have $|Qu|_{h^1} = \|\widetilde{Qu}\|_{\ell^2} = \|Q\tilde{u}\|_{\ell^2} \leq \|\tilde{u}\|_{\ell^2} = |u|_{h^1}$. □

It is worth noting that the previous argument can not be used to control $\mathbb{E}|P^{(n,m)}u|_{h^1}$: for a matrix $P = P^{(n,m)}$, the corresponding quantity $\widetilde{Pu}$ can not be easily expressed with $\tilde{u}$. Indeed, given a deterministic u, then $(P^{(n,m)}u)_j$ and $(P^{(n,m)}u)_{j+1}$ are independent random variables - since they are defined respectively with $\mathcal{N}^{(n,m,j)}$ and $\mathcal{N}^{(n,m,j+1)}$. The only result that can be proved is Proposition 4.4 below. However, its only role in the sequel is to explain why we can not obtain directly a good error bound; as a consequence, we do not give its proof.

Proposition 4.4. *There exists a constant C, such that for any discretization parameters $N \geq 1$, $\delta t = \frac{T}{M_T}$ and $\delta x = \frac{1}{M_S}$, we have for any vector $u \in V$*

$$\mathbb{E}|P^{(0)}u|^2_{h^1} \leq (1 + C\frac{\delta t + \delta x^2}{N\delta x^2})|u|^2_{h^1}. \tag{4.6}$$

Due to independence of matrices involved at different steps of the scheme, the previous inequalities can be used in chain.

We thus observe that the matrices $P^{(k)}$ and Q are quite different, even if $Q = \mathbb{E}P^{(k)}$. On the one hand, the matrix Q is symmetric, and therefore respects the structure of the heat equation - the Laplace operator is also symmetric with respect to the L^2-scalar product. On the other hand, the structure of the noise destroys this symmetry for matrices $P^{(k)}$, while it introduces many other properties due to independence - in some sense noise is white in space and implies first that solutions are not regular, but that on the average a better estimate can be obtained.

5 Proof of Theorems 2.2 and 2.3

We begin with a detailed proof of (2.5). A proof of the other part of the error (2.4) is given in Section 5.4 below. Easy computations give the following expression for the Monte-Carlo part of the error: for any $0 \leq n \leq M_T$

$$\delta x \sum_{j=0}^{M_S-1} \mathrm{Var}(u^n_j) = \delta x \sum_{j=0}^{M_S-1} \mathbb{E}|u^n_j - v^n_j|^2 = \mathbb{E}\|u^n - v^n\|^2_{\ell^2} = \delta x \mathbb{E}(u^n - v^n)^*(u^n - v^n),$$

where the superscript $*$ denotes transposition of matrices.

Using the notations introduced in Section 4, since the vectors u^n and v^n satisfy (4.5), with the same deterministic initial condition u^0, we have

$$\begin{aligned}\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 &= \mathbb{E}\|(P^{(n-1)}\dots P^{(0)} - Q^n)u^0\|_{\ell^2}^2 \\ &= \delta x (u^0)^* \mathbb{E}\left((P^{(n-1)}\dots P^{(0)} - Q^n)^*(P^{(n-1)}\dots P^{(0)} - Q^n)\right) u^0 \\ &= \delta x (u^0)^* \mathbb{E}\left((P^{(0)})^*\dots(P^{(n-1)})^* P^{(n-1)}\dots P^{(0)} - (Q^n)^* Q^n\right) u^0,\end{aligned}$$

where the last equality is a consequence of the relation $\mathbb{E}P^{(k)} = Q$ and of the independence of the matrices $P^{(k)}$.

Therefore we need to study the matrix $S_n = \mathbb{E}\left((P^{(0)})^*\dots(P^{(n-1)})^* P^{(n-1)}\dots P^{(0)} - (Q^n)^* Q^n\right)$ given by the expression above, such that

$$\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 = \delta x (u^0)^* S_n u^0.$$

The proof is now decomposed in the following steps. First, we identify two useful decompositions of the error in Section 5.1. Then in Section 5.2, we analyze the variance of the error committed at each step. Finally, proofs of the Theorems are given in Section 5.3.

5.1 Decompositions of the error

We propose two decompositions of S_n into sums of n terms, involving products of matrices $P^{(k)}$, of Q and of the difference between two matrices $P^{(k)}$ and Q, which corresponds to a one-step error:

$$P^{(n-1)}\dots P^{(0)} - Q^n = \sum_{k=0}^{n-1} P^{(n-1)}\dots P^{(k+1)}\big(P^{(k)} - Q\big)Q^k, \tag{5.1}$$

and

$$P^{(n-1)}\dots P^{(0)} - Q^n = \sum_{k=0}^{n-1} Q^{n-1-k}\big(P^{(k)} - Q\big)P^{(k-1)}\dots P^{(0)}. \tag{5.2}$$

These decompositions lead to the following expressions for S_n - where we use the independence of the matrices $P^{(k)}$ for different values of k:

$$\begin{aligned}S_n &= \mathbb{E}\sum_{k=0}^{n-1}(Q^k)^*\Big(P^{(k)} - Q\Big)^*(P^{(k+1)})^*\dots(P^{(n-1)})^* P^{(n-1)}\dots P^{(k+1)}\Big(P^{(k)} - Q\Big)Q^k \\ &= \mathbb{E}\sum_{k=0}^{n-1}(P^{(0)})^*\dots(P^{(k-1)})^*\Big(P^{(k)} - Q\Big)^*(Q^{n-1-k})^* Q^{n-1-k}\Big(P^{(k)} - Q\Big)P^{(k-1)}\dots P^{(0)}.\end{aligned}$$

Therefore we obtain the following expressions for the error:

$$\begin{aligned}\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 &= \delta x (u^0)^* S_n u^0 \\ &= \sum_{k=0}^{n-1}\mathbb{E}\|P^{(n-1)}\dots P^{(k+1)}\big(P^{(k)} - Q\big)Q^k u^0\|_{\ell^2}^2 \\ &= \sum_{k=0}^{n-1}\mathbb{E}\|Q^{n-1-k}\big(P^{(k)} - Q\big)P^{(k-1)}\dots P^{(0)}u^0\|_{\ell^2}^2.\end{aligned} \tag{5.3}$$

Before we show how each decomposition is used to obtain a convergence result, we focus on the variance induced by one step of the scheme. In fact, only the second one gives the improved estimate of Theorem 2.2. Nevertheless, we also get a useful error bound thanks to the first one; in particular, the main independence argument to get the variance reduction appears there.

5.2 One-step variance

In the previous Section, we have introduced decompositions of the error, and we observed that we need a bound on the error made after each time-step. The following Proposition states that the variance after one step of the scheme is of size δt if we consider the ℓ^2 norm, and that a residual term of size δx^2 appears due to the interpolation procedure. If we consider N independent realizations, Corollary 5.2 below states that the variance is divided by $1/N$ if we look at the full matrix of the scheme.

Proposition 5.1. *There exists a constant C, such that for any discretization parameters $\delta t = \frac{T}{M_T}$ and $\delta x = \frac{1}{M_S}$, and for any $1 \leq m \leq N$ and $0 \leq n \leq M_T - 1$, we have for any vector $u \in \mathbb{R}^{M_S}$*

$$\mathbb{E}\|(P^{(n,m)} - Q)u\|_{\ell^2}^2 \leq C(\delta t + \delta x^2)|u|_{h^1}^2. \tag{5.4}$$

Corollary 5.2. *For any $0 \leq n \leq M_T - 1$ and for any vector $u \in \mathbb{R}^{M_S}$, we have*

$$\mathbb{E}\|(P^{(n)} - Q)u\|_{\ell^2}^2 \leq C\frac{(\delta t + \delta x^2)}{N}|u|_{h^1}^2.$$

The proof of the corollary is straightforward, since $P^{(n)} = \frac{1}{N}\sum_{m=1}^N P^{(n,m)}$ with independent and identically distributed matrices $P^{(n,m)}$. However, the proof of Proposition 5.1 is very technical.

One difficulty of the proof is the dependence of the noise on the position j: for different indices j_1 and j_2, the random variables $(P^{(n,m)}u)_{j_1}$ and $(P^{(n,m)}u)_{j_2}$ are independent. To deal with this problem, for each j we introduce an appropriate auxiliary function and we analyze the error on each interval $[x_j, x_{j+1}]$ separately. We also need to take care of some regularity properties of the functions - they are H^1 functions, piecewise linear, but they are not in general of class $\mathcal{C}^1$ - in order to obtain bounds involving the h^1 and H^1 semi-norms.

Proof of Proposition 5.1. To simplify the notations, we assume that $n = 0$ and that $m = 1$ so that we only work with one matrix P with entries

$$P_{j,k} = \phi_k(x_j + B_{\delta t}^j),$$

where the B^j are independent Brownian Motions.

We define the following auxiliary periodic functions: for any $x \in \mathbb{R}$

$$V(x) = \mathbb{E}\mathcal{I}u(x + B_{\delta t}^j), \tag{5.5}$$

and for any index $0 \leq j \leq M_S - 1$

$$U^{(j)}(x) = \mathcal{I}u(x + B_{\delta t}^j). \tag{5.6}$$

We observe that since we take expectation in (5.5) the index j plays no role there. Moreover we have the following relations for any $j \in S$:

$$V(x_j) = (Qu)_j \quad \text{and} \quad U^{(j)}(x_j) = (Pu)_j, \quad \text{but} \quad U^{(j)}(x_{j+1}) \neq (Pu)_{j+1}.$$

The last relation is the reason why we need to introduce different auxiliary functions $U^{(j)}$ for each index j.

We finally introduce the following function depending on two variables: for any $0 \leq t \leq \delta t$ and $x \in \mathbb{R}$,

$$\mathcal{V}(t, x) = \mathbb{E}\mathcal{I}u(x + B_t), \tag{5.7}$$

for some standard Brownian Motion B. This function is solution of the backward Kolmogorov equation associated with the Brownian Motion, with the initial condition $\mathcal{V}(0, .) = \mathcal{I}u$, and for $t > 0$

$$\partial_t \mathcal{V} = \frac{1}{2}\partial_{xx}^2 \mathcal{V}.$$

Moreover we have $\mathcal{V}(\delta t, .) = V$.

We have the following expression for the mean-square error, integrated over an interval $[x_j, x_{j+1}]$: for any index $j \in S$

$$\int_{x_j}^{x_{j+1}} \mathbb{E}|U^{(j)}(x) - V(x)|^2 \mathrm{d}x = \int_0^{\delta t}\int_{x_j}^{x_{j+1}} \mathbb{E}|\partial_x \mathcal{V}(\delta t - s, x + B_s^j)|^2 \mathrm{d}x\mathrm{d}s. \tag{5.8}$$

The proof of this identity is as follows. First, thanks to smoothing properties of the heat semi-group, for any $t > 0$ the function $\mathcal{V}(t, .)$ is smooth. Using Itô formula, with the Brownian Motion B^j corresponding to the function $U^{(j)}$,

$$\mathrm{d}\mathcal{V}(\delta t - s, x + B_s^j) = \partial_x \mathcal{V}(\delta t - s, x + B_s^j)\mathrm{d}B_s^j,$$

for $0 \leq s \leq \delta t - \epsilon$ and for any $\epsilon \in (0, \delta t)$, and the isometry property implies

$$\mathbb{E}|\mathcal{V}(\delta t, x) - \mathcal{V}(\epsilon, x + B_{\delta t - \epsilon}^j)|^2 = \int_0^{\delta t - \epsilon} \mathbb{E}|\partial_x \mathcal{V}(\delta t - s, x + B_s^j)|^2 \mathrm{d}s.$$

We now integrate over $x \in [x_j, x_{j+1}]$. Then we pass to the limit $\epsilon \to 0$, using that $\mathcal{V}(0,.) = \mathcal{I}u$ is a piecewise linear function. Moreover, we use the identity $\mathcal{V}(\delta t,.) = V$. This concludes the proof of (5.8). We finally observe that in the right-hand side of the last equality we take expectation, so that we replace B^j with the Brownian Motion B - they are equal in law - and the Brownian Motion there does not depend on j anymore.

Summing over indices $j \in S$, we then get, thanks to an affine change of variables $y = x + B_s$

$$\begin{aligned}\sum_{j\in S}\int_{x_j}^{x_{j+1}} \mathbb{E}|U^{(j)}(x) - V(x)|^2 \mathrm{d}x &= \int_0^{\delta t}\int_0^1 \mathbb{E}|\partial_x \mathcal{V}(\delta t - s, x + B_s)|^2 \mathrm{d}x\mathrm{d}s \\ &= \int_0^{\delta t}\int_0^1 |\partial_x \mathcal{V}(\delta t - s, x)|^2 \mathrm{d}x\mathrm{d}s = \int_0^{\delta t} |\mathcal{V}(\delta t - s,.)|^2_{H^1}\mathrm{d}s \\ &\leq \int_0^{\delta t} |\mathcal{V}(0,.)|^2_{H^1}\mathrm{d}s = \delta t |\mathcal{I}u|^2_{H^1} = \delta t |u|^2_{h^1}.\end{aligned}$$

The inequality (5.4) is then a consequence of the two following estimates: first,

$$\sum_{j\in S}\int_{x_j}^{x_{j+1}} \mathbb{E}\Big|U^{(j)}(x) - V(x) - \mathcal{I}\circ\mathcal{P}(U^{(j)} - V)(x)\Big|^2 \mathrm{d}x \leq C\delta x^2 |u|^2_{h^1}, \tag{5.9}$$

and second we show that

$$\Big|\mathbb{E}\|Pu - Qu\|^2_{\ell^2} - \sum_{j\in S}\int_{x_j}^{x_{j+1}} \mathbb{E}|\mathcal{I}\circ\mathcal{P}(U^{(j)} - V)(x)|^2 \mathrm{d}x\Big| \leq C\delta x^2 |u|^2_{h^1}. \tag{5.10}$$

To get (5.9), we use the inequality (5.25) on each interval $[x_j, x_{j+1}]$, for a fixed realization of $B^j_{\delta t}$:

$$\begin{aligned}\int_{x_j}^{x_{j+1}} |U^{(j)}(x) - V(x) - \mathcal{I}\circ\mathcal{P}(U^{(j)} - V)(x)|^2 \mathrm{d}x &\leq C\delta x^2 \int_{x_j}^{x_{j+1}} |\partial_x (U^{(j)} - V)(x)|^2 \mathrm{d}x \\ &\quad + C\delta x \delta x^2 \frac{|[U^{(j)}(x_{j+1}) - V(x_{j+1})] - [U^{(j)}(x_j) - V(x_j)]|^2}{\delta x^2}.\end{aligned}$$

Taking the sum over indices $j \in S$ and expectation, we see that

$$\begin{aligned}\sum_{j\in S}\int_{x_j}^{x_{j+1}} \mathbb{E}|\partial_x (U^{(j)} - V)(x)|^2 \mathrm{d}x &\leq 2\sum_{j\in S}\int_{x_j}^{x_{j+1}} \mathbb{E}|\partial_x (\mathcal{I}u)(x + B^j_{\delta t})|^2 \mathrm{d}x + \sum_{j\in S}\int_{x_j}^{x_{j+1}} |\partial_x V(x)|^2 \mathrm{d}x \\ &\leq 2(|\mathcal{I}u|^2_{H^1} + |\mathcal{V}(\delta t,.)|^2_{H^1}) \leq 4|\mathcal{I}u|^2_{H^1} = 4|u|^2_{h^1},\end{aligned}$$

since $V = \mathcal{V}(\delta t,.)$. Indeed, taking expectation allows to consider a single Brownian Motion B, without j-dependence.

We now decompose the remaining term as follows:

$$\begin{aligned}\frac{|[U^{(j)}(x_{j+1}) - V(x_{j+1})] - [U^{(j)}(x_j) - V(x_j)]|^2}{\delta x^2} &\leq 2\frac{|U^{(j)}(x_{j+1}) - U^{(j)}(x_j)|^2}{\delta x^2} \\ &\quad + 2\frac{|V(x_{j+1}) - V(x_j)|^2}{\delta x^2}.\end{aligned}$$

With the second part, using Proposition 4.3 we see that

$$\delta x \sum_{j\in S} \frac{|V(x_{j+1}) - V(x_j)|^2}{\delta x^2} = \delta x \sum_{j\in S} \frac{|(Qu)_{j+1} - (Qu)_j|^2}{\delta x^2} = |Qu|^2_{h^1} \leq |u|^2_{h^1}.$$

To treat the first part, we make the fundamental observation that for a fixed $j \in S$, the same noise process B^j is used to compute all values $U^{(j)}(x)$ when x varies. As a consequence, we can use a pathwise, almost sure version of the argument leading to the proof of Proposition 4.3 which concerns the behavior of Q with respect to the h^1 semi-norm.

$$\begin{aligned}U^{(j)}(x_{j+1}) - U^{(j)}(x_j) &= \sum_{k\in S} u_k[\phi_k(x_{j+1} + B^j_{\delta t}) - \phi_k(x_j + B^j_{\delta t})] \\ &= \sum_{k\in S} u_k[\phi_{k-1}(x_j + B^j_{\delta t}) - \phi_k(x_j + B^j_{\delta t})] \\ &= \sum_{k\in S} [u_{k+1} - u_k]\phi_k(x_j + B^j_{\delta t}),\end{aligned}$$

using the relation $\phi_{k+1}(x) = \phi_k(x - \delta x)$ and an integration by parts.

Now summing over indices $j \in S$ and using the Jensen inequality - thanks to Proposition 4.1 - we obtain

$$\delta x \sum_{j\in S} \mathbb{E}\frac{|U^{(j)}(x_{j+1}) - U^{(j)}(x_j)|^2}{\delta x^2} \le \delta x \sum_{k\in S, j\in S} \mathbb{E}\phi_k(x_j + B^j_{\delta t})\frac{|u_{k+1} - u_k|^2}{\delta x^2}$$
$$\le \delta x \sum_{k\in S} \frac{|u_{k+1} - u_k|^2}{\delta x^2} = |u|^2_{h^1}.$$

Having proved (5.9), we now focus on (5.10). We have, since $\mathcal{P}(U^{(j)} - V)_j = [(P - Q)u]_j$

$$|\sum_{j\in S}\int_{x_j}^{x_{j+1}} |\mathcal{I}\circ\mathcal{P}(U^{(j)} - V)(x)|^2 dx - \delta x \sum_j |[(P - Q)u]_j|^2|$$
$$\le C\delta x^2 \delta x \sum_{j\in S} \frac{|(U^{(j)} - V)(x_{j+1}) - (U^{(j)} - V)(x_j)|^2}{\delta x^2}.$$

It remains to take expectation and to conclude like for (5.9).

□

5.3 Proof of the Theorems

As we have explained in the introduction, we consider that δx is controlled by δt thanks to a anti-CFL condition. Roughly, from Proposition 5.1 we thus see that the variance obtained after one step of the scheme is of size δt, and that the error depends on the solution through the h^1 semi-norm. Moreover, from Propositions 4.3 and 4.4 we remark that the behaviors of the matrices Q and $P^{(n)}$ with respect to this semi-norm are quite different.

Using the first decomposition of the error in (5.3), we use in chain the bounds given above in Propositions 4.2, 4.3 and 5.1 and Corollary 5.2:

$$\mathbb{E}\|u^n - v^n\|^2_{\ell^2} = \sum_{k=0}^{n-1} \mathbb{E}\|P^{(n-1)}\dots P^{(k+1)}(P^{(k)} - Q)Q^k u^0\|^2_{\ell^2} \le \sum_{k=0}^{n-1} \mathbb{E}\|(P^{(k)} - Q)Q^k u^0\|^2_{\ell^2}$$
$$\le \sum_{k=0}^{n-1} C\frac{(\delta t + \delta x^2)}{N}|Q^k u^0|^2_{h^1} \le \sum_{k=0}^{n-1} C\frac{(\delta t + \delta x^2)}{N}|u^0|^2_{h^1}$$
$$\le C\frac{1 + \delta x^2/\delta t}{N}|u^0|^2_{h^1}.$$

If the continuous problem is initialized with the function u_0, which is periodic and of class $\mathcal{C}^1$, then $u^0 = \mathcal{P}u_0$ satisfies $|u^0|_{h^1} \le \sup_{x\in[0,1]} |u_0'(x)|$. Moreover we assume that an anti-CFL condition is satisfied, so that the term $\delta x^2/\delta t$ is bounded. As a consequence, we find a classical Monte-Carlo estimate, where the error does not decrease when δt goes to 0 and is only controlled with the number of realizations:

$$\mathbb{E}\|u^n - v^n\|^2_{\ell^2} \le C\frac{1 + \delta x^2/\delta t}{N}|u^0|^2_{h^1}. \tag{5.11}$$

In fact, (5.11) shows that the variances obtained at each time step can be summed to obtain some control of the variance at the final time. To get an improved bound, we thus need other arguments.

The main observation is that using independence of rows in the P-matrices, we only need to focus on diagonal terms

$$\sup_{j\in S}\left((Q^\ell)^* Q^\ell\right)_{jj} = \sup_{j\in S}\left(Q^{2\ell}\right)_{jj},$$

for indices $0 \le \ell = n - k - 1 \le n - 1$. We recall that indeed Q is a symmetric matrix, so that $(Q^\ell)^* Q^\ell = Q^{2\ell}$.

More precisely, the error can be written

$$\mathbb{E}\|u^n - v^n\|^2_{\ell^2} = \delta x (u^0)^* S_n u^0 = \delta x \sum_{i,j\in S} u^0_i (S_n)_{i,j} u^0_j$$
$$= \delta x \sum_{k=0}^{n-1}\sum_{i,j\in S} u^0_i \mathbb{E}\big((A_k)^*(P^{(k)} - Q)Q^{2(n-1-k)}(P^{(k)} - Q)A_k\big)_{i,j} u^0_j,$$

where for simplicity we use the notation $A_k := P^{(k-1)} \dots P^{(0)}$. We compute for any $i, j \in S$, using the independence properties at different steps

$$\begin{aligned}
&\mathbb{E}((A_k)^*(P^{(k)} - Q)Q^{2(n-1-k)}(P^{(k)} - Q)A_k)_{i,j} \\
&\quad = \sum_{k_1,k_2,k_3,k_4 \in S} \mathbb{E}[(A_k)_{k_1,i}(P^{(k)} - Q)_{k_2,k_1}(Q^{2(n-1-k)})_{k_2,k_3}(P^{(k)} - Q)_{k_3,k_4}(A_k)_{k_4,j}] \\
&\quad = \sum_{k_1,k_2,k_3,k_4 \in S} \mathbb{E}[(A_k)_{k_1,i}(A_k)_{k_4,j}]\mathbb{E}[(P^{(k)} - Q)_{k_2,k_1}(P^{(k)} - Q)_{k_3,k_4}](Q^{2(n-1-k)})_{k_2,k_3}.
\end{aligned}$$

The observation is now that if $k_2 \neq k_3$, then the independence of the random variables for different nodes implies that

$$\mathbb{E}[(P^{(k)} - Q)_{k_2,k_1}(P^{(k)} - Q)_{k_3,k_4}] = 0, \tag{5.12}$$

since it is the covariance of two independent random variables - see (4.2). Moreover, when $k_2 = k_3$ we see that $((Q^{(n-1-k)})^* Q^{(n-1-k)})_{k_2,k_3}$ only depends on $n - k - 1$, due to invariance properties of the equation. Therefore we rewrite the former expansion in the following way:

$$\begin{aligned}
\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 &= \delta x \sum_{k=0}^{n-1} \sum_{i,j \in S} u_i^0 \mathbb{E}\big((A_k)^*(P^{(k)} - Q)Q^{2(n-1-k)}(P^{(k)} - Q)A_k\big)_{i,j} u_j^0 \\
&= \delta x \sum_{k=0}^{n-1} \sum_{i,j \in S} u_i^0 u_j^0 (Q^{2(n-1-k)})_{1,1} \sum_{k_2 \in S} \mathbb{E}\left[\Big((P^{(k)} - Q)A_k\Big)_{k_2,i} \Big((P^{(k)} - Q)A_k\Big)_{k_2,j}\right] \\
&= \sum_{k=0}^{n-1} \Big(Q^{2(n-1-k)}\Big)_{1,1} \mathbb{E}\|(P^{(k)} - Q)P^{(k-1)} \dots P^{(0)} u^0\|_{\ell^2}^2.
\end{aligned} \tag{5.13}$$

We thus have to control $(Q^{2\ell})_{1,1} = (Q^{2\ell})_{j,j}$ for any $j \in S$. The following Lemma 5.3 gives a control of this expression. The first estimate means that the coefficients $Q^{2\ell}_{j_1,j_2}$ are approximations of the solution of the PDE at time $2\ell\delta t$, at position j_2, starting from the initial condition ϕ_{j_1}, with an error due to interpolation. The second estimate is fundamental in the proof of the Theorem, since it allows to introduce an additional factor δx; however, we need to treat carefully the denominator.

Lemma 5.3. *There exists a constant C such that for any discretization parameters $\delta t = \frac{T}{M_T}$ and $\delta x = \frac{1}{M_S}$, we have for any $1 \leq \ell \leq M_T - 1$ and for any $0 \leq j_1, j_2 \leq M_S - 1$*

$$|Q^{2\ell}_{j_1,j_2} - \mathbb{E}\phi_{j_1}(x_{j_2} + B_{2\ell\delta t})| \leq C \frac{\delta x^2}{\delta t}(1 + |\log(\delta t)|). \tag{5.14}$$

Moreover, for any $j \in S$, we have for any $1 \leq \ell \leq M_T$

$$\mathbb{E}\phi_j(x_j + B_{2\ell\delta t}) \leq C \frac{\delta x}{\sqrt{2\ell\delta t}}. \tag{5.15}$$

Remark 5.4. *The singularities when $\delta t \to 0$ with a fixed δx come from the use of regularization of the heat semi-group - when we consider the ϕ_j's as initial conditions.*

For the second estimate (5.15)*, we make two important remarks. First, the constant C depends on the final time T, and we cannot directly let ℓ tend to $+\infty$: we have*

$$\lim_{\ell \to +\infty} \mathbb{E}\phi_j(x_j + B_{2\ell\delta t}) = \int_{x_j - 1/2}^{x_j + 1/2} \phi_j(x)\mathrm{d}x = \delta x \neq 0.$$

Second, from (5.15) *we get for any $\ell > 0$ and for any fixed δt*

$$\lim_{\delta x \to 0} \mathbb{E}\phi_j(x_j + B_{2\ell\delta t}) = 0,$$

while we know that for a fixed $\delta x > 0$ and a fixed ℓ, we have

$$\lim_{\delta t \to 0} \mathbb{E}\phi_j(x_j + B_{2\ell\delta t}) = \phi_j(x_j) = 1.$$

These two behaviors are different and from (5.15) *we see the kind of relations that the parameters δx and δt must satisfy for obtaining one convergence or the other.* □

Proof of Lemma 5.3. For any $0 \leq \ell \leq 2M_T$, we define

$$\mathbf{M}_\ell = \sup_{i,j \in S} |(Q^\ell)_{i,j} - \mathbb{E}\phi_j(x_i + B_{\ell\delta t})|,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian Motion.

We have $\mathbf{M}_0 = 0$, and by definition of Q we also have $\mathbf{M}_1 = 0$.

We define some auxiliary functions W_j, for any index $j \in S$: for any $x \in \mathbb{R}$ and any $t \geq 0$

$$W_j(t,x) = \mathbb{E}\phi_j(x + B_t).$$

W_j is solution of the heat equation, with periodic boundary conditions and initial condition ϕ_j. For any $t > 0$, $W_j(t,.)$ is therefore a smooth function - thanks to regularization properties of the heat semi-group - and since ϕ_j is bounded by 1 we easily see that we have the following estimates, for some constant C:

$$\|\partial_x W_j(t,.)\|_\infty \leq \frac{C}{\sqrt{t}} \quad \text{and} \quad \|\partial^2_{xx} W_j(t,.)\|_\infty \leq \frac{C}{t}. \tag{5.16}$$

We now prove the following estimate on the sequence $(\mathbf{M}_\ell)$: for any $1 \leq \ell \leq M_T - 1$

$$\mathbf{M}_{\ell+1} \leq \mathbf{M}_\ell + C\frac{\delta x^2}{\ell\delta t}. \tag{5.17}$$

The error comes from the interpolation procedure which is made at each time step.

For any $i, j \in S$, Markov property implies that

$$\begin{aligned}(Q^{\ell+1})_{i,j} - \mathbb{E}\phi_j(x_i + B_{(\ell+1)\delta t}) &= \sum_{k\in S} Q_{i,k}(Q^\ell)_{k,j} - \mathbb{E}W_j(\ell\delta t, x_i + B_{\delta t}) \\ &= \sum_{k\in S} Q_{i,k}(Q^\ell)_{k,j} - \mathbb{E}\mathcal{I}\circ\mathcal{P}(W_j(\ell\delta t,.))(x_i + B_{\delta t}) \\ &+ \mathbb{E}[\mathcal{I}\circ\mathcal{P}(W_j(\ell\delta t,.)) - W_j(\ell\delta t,.)](x_i + B_{\delta t}).\end{aligned}$$

For the first term, we remark that it is bounded by $\mathbf{M}_\ell$; indeed we see that

$$\begin{aligned}\mathbb{E}\mathcal{I}\circ\mathcal{P}(W_j(\ell\delta t,.))(x_i + B_{\delta t}) &= \sum_{k\in S} W_j(\ell\delta t, x_k)\mathbb{E}\phi_k(x_i + B_{\delta t}) \\ &= \sum_{k\in S} Q_{i,k}\mathbb{E}\phi_j(x_k + B_{\ell\delta t}).\end{aligned}$$

To conclude, it remains to use the stochasticity of the matrix Q: entries are positive, and their sum over each line is equal to 1.

The second term is bounded using the following argument:

$$\|\mathcal{I}\circ\mathcal{P}(W_j(\ell\delta t,.)) - W_j(\ell\delta t,.)\|_\infty \leq C\delta x^2 \|\partial^2_{xx} W_j(\ell\delta t,.)\|_\infty \leq C\frac{\delta x^2}{\ell\delta t},$$

according to well-known interpolation estimates and to (5.16).

From (5.17), using $\mathbf{M}_1 = 0$ we obtain for any $1 \leq \ell \leq M_T$

$$\mathbf{M}_\ell \leq C\frac{\delta x^2}{\delta t}\sum_{k=1}^{M_T-1}\frac{1}{k} \leq C\frac{\delta x^2}{\delta t}(|\log(T)| + |\log(\delta t)|),$$

which gives the result, with a constant depending on T.

Now we prove the second estimate of the Lemma. Thanks to the relation $\phi_{k+1}(x) = \phi_k(x - \delta x)$ we see that the left-hand side does not depend on $j \in S$; moreover we expand the calculation of the expectation using the periodicity of the function ϕ_j and relation definition of $\hat{\phi}$, the description of its support as

$x_j + \bigcup_{k\in\mathbb{Z}}[k-\delta x, k+\delta x]$: we get for $1 \le \ell \le M_T$

$$\begin{aligned}
\mathbb{E}\phi_j(x_j + B_{2\ell\delta t}) &= \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{2\pi\ell\delta t}} \int_{k-\delta x}^{k+\delta x} \hat{\phi}(\frac{z-k}{\delta x}) e^{-|z|^2/(2\ell\delta t)} \mathrm{d}z \\
&\le \frac{1}{\sqrt{2\pi\ell\delta t}} \sum_{k\in\mathbb{Z}} \int_{k-\delta x}^{k+\delta x} e^{-|z|^2/(2\ell\delta t)} \mathrm{d}z \\
&\le \frac{1}{\sqrt{2\pi\ell\delta t}} \sum_{k\in\mathbb{Z}} \int_{k-\delta x}^{k+\delta x} e^{-|z|^2/(2T)} \mathrm{d}z \\
&\le \frac{1}{\sqrt{2\pi\ell\delta t}} \sum_{k\in\mathbb{Z}} C\delta x e^{-k^2/(2T)} \\
&\le C\frac{\delta x}{\sqrt{2\ell\delta t}}.
\end{aligned}$$

□

The estimate of Lemma 5.3 is now used in (5.13), and we obtain:

$$\begin{aligned}
\sum_{k=0}^{n-1} \left(Q^{2(n-1-k)}\right)_{1,1} &\le C \sum_{k=0}^{n-2} \left(\frac{\delta x^2}{\delta t}(1 + |\log(\delta t)|) + \frac{\delta x}{\sqrt{(n-1-k)\delta t}} \right) + 1 \\
&\le C\left(\frac{\delta x^2}{\delta t^2}(1 + |\log(\delta t)|) + \frac{\delta x}{\delta t} + 1 \right) =: C\mathcal{A}.
\end{aligned} \tag{5.18}$$

To conclude one more argument is necessary: we need to apply Proposition 5.1 in order to sum the variances. However this involves the quantity $\mathbb{E}|P^{(k-1)}\dots P^{(0)}u^0|_{h^1}^2$, which is badly controlled according to Proposition 4.4: for example, when $\delta t = \delta x$ the accumulation only implies that

$$\mathbb{E}|P^{(k-1)}\dots P^{(0)}u|_{h^1}^2 \le (1 + C\frac{\delta t}{N\delta x^2})^k |u|_{h^1}^2 \le e^{\frac{CT}{N\delta x^2}} |u|_{h^1}^2,$$

for any $k \le \frac{T}{\delta t}$. We recall that this bad behavior of the matrices $P^{(n)}$ with respect to the h^1 semi-norm is a consequence of the independence of the random variables for different nodes, whereas this independence property is essential to get the improved estimate, since it allows to use the second estimate of Lemma 5.3.

Remark 5.5. *Instead of considering Gaussian random variables which are independent with respect to the spatial index j, we could more generally introduce - like in [7] - a correlation matrix K, and try to minimize the variance with respect to the choice of K. Here we have chosen K as the identity matrix, so that the noise is white in space; the error bound* (2.5) *we obtain is a nontrivial consequence of an averaging effect due to this choice - see* (5.12)*. A natural question - which is not answered here - would be to analyze the situation for general K: do we still improve the variance, and can we get more regular solutions?*

The solution we propose relies on the following idea: if above we could replace $P^{(k-1)}\dots P^{(0)}$ with Q^k, we could easily conclude. Another error term appears, which is controlled by $1/N$ instead of $1/\sqrt{N}$. More precisely, independence properties yield for $k \ge 1$

$$\begin{aligned}
\mathbb{E}\|(P^{(k)} - Q)P^{(k-1)}\dots P^{(0)}u^0\|_{\ell^2}^2 &= \mathbb{E}\|(P^{(k)} - Q)Q^k u^0\|_{\ell^2}^2 \\
&\quad + \mathbb{E}\|(P^{(k)} - Q)\left(P^{(k-1)}\dots P^{(0)} - Q^k\right)u^0\|_{\ell^2}^2.
\end{aligned} \tag{5.19}$$

The roles of the different terms are as follows. On the one hand, the first term gives the part of size $\frac{\delta t}{N}$, thanks to Lemma 5.3: according to Corollary 5.2 and to Proposition 4.3, we have for any $k \ge 1$ with $k\delta t \le T$

$$\mathbb{E}\|(P^{(k)} - Q)Q^k u^0\|_{\ell^2}^2 \le C\frac{\delta t + \delta x^2}{N}|Q^k u^0|_{h^1}^2 \le C\frac{\delta t + \delta x^2}{N}|u^0|_{h^1}^2. \tag{5.20}$$

On the other hand, the second term is now used to improve recursively the error estimate, since we have

$$\mathbb{E}\|(P^{(k)} - Q)\left(P^{(k-1)}\dots P^{(0)} - Q^k\right)u^0\|_{\ell^2}^2 \le \frac{C}{N}\mathbb{E}\|\left(P^{(k-1)}\dots P^{(0)} - Q^k\right)u^0\|_{\ell^2}^2. \tag{5.21}$$

The independence of realizations at step k gives the factor $\frac{1}{N}$; we remark that we cannot use the estimation of the one-step variance given by Corollary 5.2: otherwise we would need to control $\mathbb{E}\| \left(P^{(k-1)} \dots P^{(0)} - Q^k\right) u^0\|_{h^1}^2$.

Using also (5.18) and (5.19) into (5.13), we see that

$$\sup_{n\in\mathbb{N},n\delta t\leq T} \mathbb{E}\|u^n - v^n\|_{\ell^2}^2 \leq C\delta t\mathcal{A}\frac{1+\frac{\delta x^2}{\delta t}}{N}|u^0|_{h^1}^2 + C\frac{\mathcal{A}}{N}\sup_{n\in\mathbb{N},n\delta t\leq T} \mathbb{E}\|u^n - v^n\|_{\ell^2}^2. \tag{5.22}$$

The proof of the Theorem 2.2 now reduces to the study of the following recursive inequalities, for $p \geq 0$

$$E^{(p+1)} \leq C\delta t\mathcal{A}\frac{1+\frac{\delta x^2}{\delta t}}{N}|u^0|_{h^1}^2 + \frac{C\mathcal{A}}{N}E^{(p)},$$

with an initialization $E^{(0)} = C\frac{\mathcal{B}}{N}$, according to (5.11), with the notation $\mathcal{B} := (1+\frac{\delta x^2}{\delta t})|u^0|_{h^1}^2$. We remark that the control of the matrices $P^{(k)}$ and Q with respect to the ℓ^2-norm leads to another possibility for the initialization: $E^{(0)} = 2\|u^0\|_{\ell^2}^2$; we observe that the recursion then yields the same kind of estimate.

We finally easily prove that for any $p \geq 0$ there exists a constant $C_p \geq 1$ such that

$$\sup_{n\in\mathbb{N},n\delta t\leq T} \mathbb{E}\|u^n - v^n\|_{\ell^2}^2 \leq C_p\left(\frac{\mathcal{A}^p\mathcal{B}}{N^{p+1}} + \mathcal{A}\mathcal{B}\frac{\delta t}{N}\right), \tag{5.23}$$

and the proof of Theorem 2.2 is finished.

The Theorem 2.3 is a simple consequence of the inequality (5.22) when we assume N sufficiently large, namely $C\frac{\mathcal{A}}{N} \leq \frac{1}{2}$.; we recall that the constant C above does not depend on δt, δx and N.

Remark 5.6. *If we consider the equation $\frac{\partial u}{\partial t} = \frac{\nu}{2}\frac{\partial^2 u}{\partial x^2}$ with a viscosity parameter $\nu > 0$, the quantities $\mathcal{A}$ and $\mathcal{B}$ appearing in the proof are transformed into*

$$\mathcal{A}_\nu = (1 + \frac{\delta x}{\sqrt{\nu}\delta t} + \frac{\delta x^2}{\nu\delta t^2}(1 + |\log(\delta t)|)) \quad \textit{and} \quad \mathcal{B}_\nu = (\nu + \frac{\delta x^2}{\delta t})|u^0|_{h^1}^2,$$

where the constant C does not depend on ν.

The first change in the proof concerns the analysis of the one-step variance: in (5.4)*, the right-hand side is replaced by $C(\nu\delta t + \delta x^2)$. We observe that the error due to interpolation remains the same.*

The second change concerns Lemma 5.3, where we use some regularization properties thanks to Gaussian noise: when ν goes to 0 the estimates degenerates.

As a consequence, we may observe that the estimate (2.5) *gives a bound valid for a fixed value of ν, while* (5.11) *becomes more interesting when ν is small compared with the discretization parameters.*

5.4 Accumulation of the interpolation error

To obtain Theorem 2.2, it remains to control the deterministic part of the error of the scheme, without the discretization of the expectation with the Monte-Carlo method. We thus need to prove (2.4):

for any $n \in \mathbb{N}$ such that $n\delta t \leq T$, and for any $j \in \mathbb{N}$ with $0 \leq x_j = j\delta x < 1$, we have

$$|u(n\delta t, x_j) - v_j^n| \leq C\frac{\delta x^2}{\delta t}\sup_{x\in[0,1]}|u_0''(x)|, \tag{5.24}$$

where u is the exact solution and where v^n is defined by (2.3).

Since $\|u(n\delta t, x.) - v^n\|_{\ell^2} \leq \sup_j |u(n\delta t, x_j) - v_j^n|$, we easily obtain an estimate in the ℓ^2-norm. Therefore, the conditions imposed on δx and δt by (2.5) are not restrictive, and can be seen as consequences of the semi-lagrangian framework.

The proof of (5.24) in our context is as follows: using the exact representation formula and its discrete counterpart (2.3), we have

$$\begin{aligned} u((n+1)\delta t, x_j) - v_j^{n+1} &= \mathbb{E}u(n\delta t, x_j + B_{\delta t}) - \mathbb{E}\sum_{k\in S} v_k^n\phi_k(x_j + B_{\delta t}) \\ &= \sum_{k\in S}(u(n\delta t, x_k) - v_k^n)\mathbb{E}\phi_k(x_j + B_{\delta t}) \\ &\quad + \mathbb{E}\Big(u(n\delta t, x_j + B_{\delta t}) - \sum_{k\in S} u(n\delta t, x_k)\phi_k(x_j + B_{\delta t})\Big), \end{aligned}$$

where $B_{\delta t}$ is a Brownian Motion at time δt.

It is easy to see that

$$|\sum_{k\in S}(u(n\delta t,x_k)-v_k^n)\mathbb{E}\phi_k(x_j+B_{\delta t})|\leq \sup_{k\in S}|u(n\delta t,x_k)-v_k^n|,$$

and we see that the other term depends on the interpolation error:

$$\begin{aligned}|\mathbb{E}[u(n\delta t,x_j+B_{\delta t})-\sum_{k\in S}u(n\delta t,x_k)\phi_k(x_j+B_{\delta t})]|&\leq \sup_{x\in[0,1]}|u(n\delta t,x)-\mathcal{I}\circ\mathcal{P}u(n\delta t,.)(x)|\\&\leq C\delta x^2\sup_{x\in[0,1]}|\frac{\partial^2 u}{\partial x^2}(n\delta t,x)|\leq C\delta x^2\sup_{x\in[0,1]}|\frac{\partial^2 u}{\partial x^2}(0,x)|.\end{aligned}$$

To conclude, we remark that for $n=0$ we have $u(0,x_j)=v_j^0$.

Appendix: Proof of Proposition 2.1

The first equality follows from a direct computation.

We introduce the notation

$$< f_1, f_2 >_{L^2}= \int_0^1 f_1(x)f_2(x)dx$$

for the L^2-scalar product of two square integrable functions $f_1, f_2 \in L^2(0,1)$.

The second equality is proved expanding $\mathcal{I}u=\sum_k u_k\phi_k$ with respect to the L^2 scalar product $< .,. >_{L^2}$, and rewriting the sums in order to make the h^1 semi-norm appear: we have

$$\|\mathcal{I}u\|_{L^2}^2=\|\sum_{k\in S}u_k\phi_k\|_{L^2}^2=\sum_{k,\ell\in S}u_ku_\ell\left\langle\phi_k,\phi_\ell\right\rangle_{L^2}=\frac{2\delta x}{3}\sum_{k\in S}u_k^2+\frac{\delta x}{6}\sum_{k\in S}(u_ku_{k+1}+u_ku_{k-1}),$$

where we define $u_{M_S}=u_0$ and $u_{-1}=u_{M_S-1}$, that is we extend $u\in V$ by periodicity. We also used the fact that for all k,

$$\left\langle\phi_k,\phi_\ell\right\rangle_{L^2}=0 \text{ if } \ell\notin\{k-1,k,k+1\},\quad \left\langle\phi_k,\phi_k\right\rangle_{L^2}=\frac{2\delta x}{3},\quad \text{and}\quad \left\langle\phi_k,\phi_{k-1}\right\rangle_{L^2}=\frac{\delta x}{6}.$$

Now, the equality contains $\|u\|_{h^1}^2$ which appears with natural integration by parts - using periodicity:

$$\begin{aligned}\|u\|_{\ell^2}^2-\|\mathcal{I}u\|_{L^2}^2&=\frac{\delta x}{6}\sum_{k\in S}(u_k(u_k-u_{k-1})+u_k(u_k-u_{k+1}))\\&=\frac{\delta x}{6}\sum_{k\in S}(u_{k+1}(u_{k+1}-u_k)+u_k(u_k-u_{k+1}))\\&=\frac{\delta x}{6}\sum_{k\in S}(u_{k+1}-u_k)^2=\frac{1}{6}\delta x^2\|u\|_{h^1}^2.\end{aligned}$$

To prove the last estimate, we have for any $j\in S=\{0,1,\dots M_S-1\}$ and for any $x\in[x_j,x_{j+1}]$,

$$\begin{aligned}|f(x)-(\mathcal{I}\circ\mathcal{P}f)(x)|^2&\leq 2(\int_{x_j}^x f'(t)\mathrm{d}t)^2+2(\int_{x_j}^x\frac{f(x_{j+1})-f(x_j)}{\delta x}\mathrm{d}t)^2\\&\leq 2(x-x_j)\int_{x_j}^{x_{j+1}}|f'(t)|^2\mathrm{d}t+2\delta x^2\frac{|f(x_{j+1})-f(x_j)|^2}{\delta x^2},\end{aligned}$$

using the Cauchy-Schwarz inequality. Now we integrate over $[x_j,x_{j+1}]$, and then it remains to take the sum over $j\in S$ of the following quantities:

$$\int_{x_j}^{x_{j+1}}|f(x)-(\mathcal{I}\circ\mathcal{P}f)(x)|^2\mathrm{d}x\leq\delta x^2\int_{x_j}^{x_{j+1}}|f'(x)|^2\mathrm{d}x+2\delta x^2\frac{|f(x_{j+1})-f(x_j)|^2}{\delta x^2}\delta x. \tag{5.25}$$

The first term of the right-hand side is controlled with $|f|_{H^1}^2$, while the second term involves $|\mathcal{P}f|_{h^1}^2=|(\mathcal{I}\circ\mathcal{P})f|_{H^1}^2$. □

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